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3 (Sem–3/CBCS) MAT HC 2

2021

(Held in 2022)

MATHEMATICS

(Honours)

Paper : MAT–HC–3026

(Group Theory–I)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×10=10

(a) Give the condition on n under which the set $\{1, 2, 3, ..., n-1\}, n>1$ is a group under multiplication modulo n.

(b) Define a binary operation on the set $\mathbb{R}^n = \{(a_1, a_2, ..., a_n) : a_1, a_2, ..., a_n \in \mathbb{R}\}$ for which it is a group.

Contd.

(c) What is the centre of the dihedral group of order $2n$?

(d) Write the generators of the cyclic group $\mathbb{Z}$ (the group of integers) under ordinary addition.

(e) Show by an example that the decomposition of a permutation into a product of 2-cycles is not unique.

(f) Find the cycles of the permutation :

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 1 & 2 \end{pmatrix}$$

(g) Find the order of the permutation :

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 1 & 3 \end{pmatrix}$$

(h) Let G be the multiplicative group of all non-singular $n \times n$ matrices over $\mathbb{R}$ and let $\mathbb{R}^*$ be the multiplicative group of all non-zero real numbers. Define a homomorphism from G to $\mathbb{R}^*$.

(i) What do you mean by an isomorphism between two groups ?

(j) State the second isomorphism theorem.

2. Answer the following questions : $2 \times 5 = 10$

(a) Let G be a group and $a \in G$. Show that $\langle a \rangle$ is a subgroup of G.

(b) If G is a finite group, then order of any element of G divides the order of G. Justify whether this statement is true or false.

(c) Show that a group of prime order cannot have any non-trivial subgroup. Is it true for a group of finite composite order ?

(d) Consider the mapping ϕ from the group of real numbers under addition to itself given by $\phi(x) = [x]$, the greatest integer less than or equal to x. Examine whether ϕ is a homomorphism.

(e) Let ϕ be an isomorphism from a group G onto a group H. Prove that ϕ^{-1} is also an isomorphism from H onto G.

3. Answer the following questions : 5×4=20

(a) Show that a finite group of even order has *at least one* element of order 2.

Or

Let N be a normal subgroup of a group G. Show that G/N is abelian if and only if for all $x, y \in G$, $xyx^{-1}y^{-1} \in N$.

(b) Show that if a cyclic subgroup K of a group G is normal in G, then every subgroup of K is normal in G.

Or

Show that converse of Lagrange's theorem holds in case of finite cyclic groups.

(c) Consider the group $G = \{1, -1\}$ under multiplication. Define $f : \mathbb{Z} \to G$ by

$$f(x) = 1, \text{ if } n \text{ is even}$$
$$= -1, \text{ if } n \text{ is odd}$$

Show that f is a homomorphism from $\mathbb{Z}$ to G.

(d) Let $f : G \to G'$ be a homomorphism. Let $a \in G$ be such that $o(a) = n$ and $o(f(a)) = m$. Prove that $o(f(a)) / o(a)$, and if f is one-one, then $m = n$.

4. Answer the following questions : 10×4=40

(a) Let G be a group and $x, y \in G$ be such that $xy^2 = y^3x$ and $yx^2 = x^3y$. Then show that $x = y = e$, where e is the identity element of G. 10

Or

Give an example to show that the product of two subgroups of a group is not a subgroup in general. Also show that if H and K are two subgroups of a group G, then HK is a subgroup of G if and only if $HK = KH$. 2+8=10

(b) Prove that the order of a cyclic group is equal to the order of its generator. 10

Or

Let H be a non-empty subset of a group G. Define $H^{-1} = \left\{h^{-1} \in G : h \in H\right\}$. Show that

(i) if H is a subgroup of G, then $HH = H,\ H = H^{-1}$ and $HH^{-1} = H$;

(ii) if H and K are subgroups of G, then $(HK)^{-1} = K^{-1}H^{-1}$. 5+5=10

(c) Let G be a group and $Z(G)$ be the centre of G. If $G/Z(G)$ is cyclic, then show that G is abelian. 10

Or

State and prove Lagrange's theorem. 10

(d) Let H and K be two normal subgroups of a group G such that $H \subseteq K$. Show that $G/K \cong \frac{G/H}{K/H}$. 10

Or

Prove Cayley's theorem. 10