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3 (Sem–5/CBCS) MAT HC 1

2021

(Held in 2022)

MATHEMATICS

(Honours)

Paper : MAT–HC–5016

(Riemann Integration and Metric Spaces)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed : 1×10=10

   (a) Describe an open ball in the discrete metric space.

   (b) Find the derived set of the sets $(0,1]$ and $[0,1]$.

   (c) A subset B of a metric space (X,d) is open if and only if

   (i) $B = \overline{B}$

   (ii) $B = B^o$

   (iii) $B \neq \overline{B}$

   (iv) $B \neq B^o$

   (Choose the correct one)

Contd.

(d) Which of the following is false ?

(i) $\phi^o = \phi$, $X^o = X$

(ii) $A \subseteq B \Rightarrow A^o \subseteq B^o$

(iii) $(A \cap B)^o = A^o \cap B^o$

(iv) $(A \cup B)^o = A^o \cup B^o$

where A, B are subsets of a metric space (X, d). *(Choose the false one)*

(e) The closure of the subset

$F = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ of the real line $\mathbb{R}$ is

(i) ϕ

(ii) F

(iii) $F \cup \{0\}$

(iv) $F - \{0\}$

(Choose the correct one)

(f) In a metric space an arbitrary union of closed sets need not be closed. Justify it with an example.

(g) If A is a subset of a metric space (X, d), then which one is true ?

(i) $d(A) = d(\overline{A})$

(ii) $d(A) \neq d(\overline{A})$

(iii) $d(A) > d(\overline{A})$

(iv) $d(A) < d(\overline{A})$

(Choose the true one)

(h) When is an improper Riemann integral said to be convergent ?

(i) Evaluate $\int_0^{\infty} e^{-x} dx$ if it exists

(j) Show that $\Gamma(1) = 1$

2. Answer the following questions : 2×5=10

(a) Let F be a subset of a metric space (X, d). Prove that the set of limit points of F is a closed subset of (X, d).

(b) If F_1 and F_2 are two subsets of a metric space (X, d), then $\overline{F_1 \cap F_2} = \overline{F_1} \cap \overline{F_2}$. Justify whether it is false *or* true.

(c) Let (X, d_X) and (Y, d_Y) be metric spaces and $f: X \to Y$. If for all subsets A of X, $f(\overline{A}) \subseteq \overline{f(A)}$, then show that f is continuous on X.

(d) Let $f:[a, b] \to \mathbb{R}$ be integrable. Show that $|f|$ is integrable.

(e) Show that the function $f:[a, b] \to \mathbb{R}$ defined by $f(x) = c$ for all $x \in [a, b]$ is integrable with its integral $c(b-a)$.

3. Answer ***any four*** parts : 5×4=20

(a) Define a complete metric space. Show that the metric space $X = \mathbb{R}^n$ with the metric given by

$$d_p(x, y) = \left(\sum |x_i - y_i|^p\right)^{\frac{1}{p}}, \quad p \geq 1$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are in $\mathbb{R}^n$, is a complete metric space. 1+4=5

(b) Let (X, d_X) and (Y, d_Y) be metric spaces. Prove that a mapping $f : X \to Y$ is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subsets G of Y. 5

(c) Prove that if the metric space (X, d) is disconnected, then there exists a continuous mapping of (X, d) onto the discrete two-element space (X_0, d_0). 5

(d) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that f is integrable. 5

(e) Discuss the convergence of the integral $\int_1^\infty \frac{1}{x^p} dx$ for various values of p. 5

(f) Show that for $a > -1$,

$$S_n = \frac{1^n + 2^n + \ldots.. + n^n}{n^{1+a}} \to \frac{1}{1+a}.$$ 5

4. Answer ***any four*** parts : 10×4=40

(a) (i) Let (X, d) be a metric space. Define $d : X \times X \to \mathbb{R}$ by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} \text{ for all}$$

$x, y \in X$. Prove that d' is a metric on X.

Also show that d and d' are equivalent metrices on X. 4+2=6

(ii) Prove that a convergent sequence in a metric space is a Cauchy sequence. 4

(b) (i) Let (X, d) be a metric space and F be a subset of X. Prove that F is closed in X if and only if F^c is open. 5

(ii) If (Y, d_Y) is a subspace of a metric space (X, d), then show that a subset Z of Y is open in Y if and only if there exists an open set $G \subseteq X$ such that $Z = G \cap Y$. 5

(c) Prove that a metric space (X, d) is complete if and only if for every nested sequence $\{F_n\}_{n\geq 1}$ of non-empty closed subsets of X such that $d(F_n) \to 0$ as $n \to \infty$, the intersection $\bigcap_{n=1}^{\infty} F_n$ contains one and only one point. 10

(d) (i) Prove that in a metric space (X, d), each open ball is an open set. 4

(ii) Let (X, d_X) and (Y, d_Y) be metric spaces and $A \subseteq X$. Prove that a function $f : A \to Y$ is continuous at $a \in A$ if and only if whenever a sequence $\{x_n\}$ in A converges to a, the sequence $\{f(x_n)\}$ converges to $f(a)$. 6

(e) (i) Define uniformly continuous mapping in a metric space. Give an example to show that a continuous mapping need not be uniformly continuous. 1+4=5

(ii) Prove that the image of a Cauchy sequence under a uniformly continuous mapping is itself a Cauchy sequence. 5

(f) Let $(\mathbb{R}, d)$ be the space of real numbers with the usual metric. Prove that a subset $I \subseteq \mathbb{R}$ is connected if and only if I is an interval. 10

(g) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Show that f is integrable if and only if it is Riemann integrable. 10

(h) (i) State and prove first fundamental theorem of calculus. Using it show that

$$\int_0^a f(x)dx = \frac{a^4}{4} \quad \text{for} \quad f(x) = x^3.$$

1+3+2=6

(ii) Let f be continuous on $[a, b]$. Prove that there exists $c \in [a, b]$ such that $\frac{1}{b-a}\int_a^b f(x)dx = f(c)$.

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