Total number of printed pages–32

3 (Sem–5/CBCS) MAT HE 1/HE 2/HE 3

2022

MATHEMATICS

(Honours Elective)

Answer the Questions from any one Option.

OPTION–A

Paper : MAT–HE–5016

(Number Theory)

DSE (H)–1

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

PART–A

1. Choose the correct option in each of the following questions : ***(any ten)*** $1 \times 10 = 10$

   (i) Number of integers which are less than and co-prime to 108 is

      (a) 18

      (b) 17

Contd.

(c) 15

(d) 36

(ii) The number of positive divisors of a perfect square number is

(a) odd

(b) even

(c) prime

(d) Can't say

(iii) If $100! \equiv x(mod\,101)$, then x is

(a) 99

(b) 100

(c) 101

(d) None of the above

(iv) The solution of pair of linear congruences $2x \equiv 1\,(mod\ 5)$ and $x \equiv 4\ (mod\ 3)$ is

(a) $x \equiv 13(mod\,5)$

(b) $x \equiv 28(mod\,5)$

(c) $x \equiv 13(mod\,15)$

(d) $x \equiv 13(mod\,3)$

(v) If $a = qb$ for some integer q and $a, b \neq 0$, then

(a) b divides a

(b) a divides b

(c) $a = b$

(d) None of the above

(vi) If a and b are any two integers, then there exists some integres x and y such that

(a) $gcd(a,b) = ax + by$

(b) $gcd(a,b) = ax - by$

(c) $gcd(a,b) = ax^n + by^m$

(d) $gcd(a,b) = (ax + by)^n$

(vii) The linear diophantine equation $ax + by = c$ with $d = gcd(a,b)$ has a solution in integers if and only if

(a) $d \mid c$

(b) $c \mid d$

(c) $d \mid (ax+by)$

(d) Both (a) and (c)

(viii) If a positive integer n divides the difference of two integers a and b, then

(a) $a \equiv b \pmod{n}$

(b) $a = b \pmod{n}$

(c) $a \equiv n \pmod{b}$

(d) None of the above

(ix) The set of integers such that every integer is congruent modulo m to exactly one integer of the set is called _____ modulo m. *(Fill in the blank)*

(a) Reduced residue system

(b) Complete residue system

(c) Elementary residue system

(d) None of the above

(x) Which of the following statement is false ?

(a) There is no pattern in prime numbers

(b) No formulae for finding prime numbers

(c) Both *(a)* and *(b)*

(d) None of the above

(*xi*) The reduced residue system is ____ of complete residue system.

(*a*) compliment

(*b*) subset

(*c*) not a subset

(*d*) Both (*a*) and (*c*)

(*xii*) The unit place digit of 137^{93} is

(*a*) 7

(*b*) 9

(*c*) 3

(*d*) 1

(*xiii*) Euler phi-function of a prime number p is

(*a*) p

(*b*) $p-1$

(*c*) $p/2-1$

(*d*) None of the above

(xiv) Which theorem states that "if p is prime, then $(p-1)! \equiv -1 (mod\, p)$" ?

(a) Dirichlet's theorem

(b) Wilson's theorem

(c) Euler's theorem

(d) Fermat's little theorem

(xv) Let p be an odd prime. Then $x^2 \equiv -1 (mod\, p)$ has a solution if p is of the form

(a) $4k+1$

(b) $4k$

(c) $4k+3$

(d) None of the above

(xvi) Let m be a positive integer. Two integers a and b are congruent modulo m if and only if

(a) $m \mid (a-b)$

(b) $m \mid (a+b)$

(c) $m \mid (ab)$

(d) Both (b) and (c)

(xvii) If $ac \equiv bc(\bmod m)$ and $d = \gcd(m,c)$

(a) $a \equiv b\left(\bmod \frac{m}{d}\right)$

(b) $a \equiv c\left(\bmod \frac{m}{d}\right)$

(c) $a \equiv m(\bmod b)$

(d) $a \equiv m\left(\bmod \frac{b}{a}\right)$

(xviii) If a is a whole number and p is a prime number, then according to Fermat's theorem

(a) $a^p - a$ is divisible by p

(b) $a^p - 1$ is divisible by p

(c) $a^{p-1} - 1$ is divisible by p

(d) $a^{p-1} - a$ is divisible by p

2. Answer ***any five*** questions : $2 \times 5 = 10$

(a) Find last two digits of 3^{100} in its decimal expansion.

(b) If p and q are positive integers such that $gcd(p,q)=1$, then show that $gcd(a+b, a-b)=1$ *or* 2.

(c) Find the solution of the following linear Diophantine equation $8x-10y=42$.

(d) If p and q are any two real numbers, then prove that $[p]+[q]\leq[p+q]$ (where $[x]$ denotes the greatest integer less *or* equal to x).

(e) If m and n are integers such that $(m,n)=1$, then $\varphi(mn)=\varphi(m)\varphi(n)$.

(f) Find (7056).

(g) If $a\equiv b(mod\,n)$ and $m\mid n$, then show that $a\equiv b(mod\,m)$.

(h) List all primitive roots modulo 7.

(i) If $n=p_1^{k_1}p_2^{k_2}....p_m^{k_m}$ is the prime factorization of $n>1$, then prove that $\tau(n)=(k_1+1)(k_2+1)........(k_n+1)$.

(j) Evaluate the exponent of 7 in 1000 !

3. Answer ***any four*** questions : 5×4=20

(a) If p is a prime, then prove that

$$\varphi(p!) = (p-1)\varphi((p-1)!)$$

(b) Show that, the set of integers $\{1,5,7,11\}$ is a reduced residue system (RRS) modulo 12.

(c) Solve the following simultaneous congruence :

$$x \equiv 2(mod 3)$$
$$x \equiv 2(mod 2)$$
$$x \equiv 3(mod 5)$$

(d) For $n = p^k, p$ is a prime, prove that $n = \sum_{d|n} \varphi(d)$, where $\sum_{d|n}$ denotes the sum over all positive divisors of n.

(e) If p_n is the n^{th} prime, then show that $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_n}$ is not an integer.

(f) Let n be any integer > 2. Then $\varphi(n)$ is even.

(g) Show that if $a_1, a_2, \ldots, a_{\varphi(m)}$ is a RRS modulo m, where m is a positive integer with $m \neq 2$, then

$a_1 + a_2 + \ldots + a_{\varphi(m)} \equiv 0(mod\, m)$.

(h) Show that $10! + 1$ is divisible by 11.

PART–B

Answer ***any four*** of the following questions : $10\times4=40$

4. (a) If $a, b \neq 0$ and c be any three integers and $d = gcd(a,b)$. Then show that $ax + by = c$ has a solution iff $d \mid c$.

   Furthermore, show that if x_0 and y_0 is a particular solution of $ax + by = c$, then any other solution of the equation is $x' = x_0 - \frac{b}{d}t$ and $y' = y_0 + \frac{a}{d}t$, t is an integer. 7

   (b) Find the general solution of $10x - 8y = 42; x, y \in Z$ 3

5. (a) Show that an odd prime p can be represented as sum of two squares iff $p \equiv 1 (mod 4)$. 7

   (b) Find all positive solutions of $x^2 + y^2 = z^2$, where $0 < z < 30$. 3

6. *(a)* Prove that the quadratic congruence $x^2 + 1 \equiv 0 \pmod{p}$, where p is an odd prime, has a solution iff $p \equiv 1 \pmod{4}$. 5

(b) If p is prime and a is an integer not divisible by p, prove that $a^{p-1} \equiv 1 \pmod{p}$. 5

7. State and prove Chinese remainder theorem. Also find all integers that leave a remainder of 4 when divided by 11 and leaves a remainder of 3 when divided by 17.

8. *(a)* For each positive integer $n \geq 1$, show that $\sum_{d|n} \mu(d) = \begin{cases} 1, \text{ if } n = 1 \\ 0, \text{ if } n > 1 \end{cases}$ 5

(b) If k denotes the number of distinct prime factors of positive integer n. Prove that $\sum_{d|n} |\mu(d)| = 2^k$ 5

9. *(a)* If p is a prime, prove that $\varphi(p^k) = p^k - p^{k-1}$, for any positive integer k. For $n > 2$, show that $\varphi(n)$ is an even integer. 3+2=5

(b) State Mobius inversion formula. If the integer $n > 1$ has the prime factorization. If $n = p_1^{k_1} p_2^{k_2} \ldots. p_s^{k_s}$, then prove that

$$\sum_{d|n} \mu(d)\,\sigma(d) = (-1)^s p_1 p_2 \cdots p_s.$$ 5

10. If x be any real number. Then show that 1+3+3+3=10

(a) $[x] \leq x < [x] + 1$

(b) $[x + m] = [x] + m$, m is any integer

(c) $[x] + [-x] = \begin{cases} 0, & \text{if } x \text{ is an integer} \\ -1, & \text{otherwise} \end{cases}$

(d) $\left[\dfrac{[x]}{m}\right] = \left[\dfrac{x}{m}\right]$, if m is a positive integer

11. (a) If $a_1, a_2, \ldots, a_m$ is a complete residue system modulo m, and if k is a positive integer with $(k, m) = 1$ then $ka_1 + b,\ ka_2 + b,\ \ldots,\ ka_m + b,$ is a complete residue system modulo m for any integer b. 5

(b) Examine whether the following set forms a complete residue system or a reduced residue system :

$$\{-3,14,3,12,37,56,-1\}(mod 7)$$ 5

12. (a) If $n \geq 1$ is an integer then show that

$$\Pi_{d|n} d = n^{\frac{\tau(n)}{2}}$$ 3

(b) If f and g are two arithmetic functions, then show that the following conditions are equivalent : 7

(i) $f(n) = \sum_{d|n} g(d)$

(ii) $g(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d)$

13. (a) If n is a positive integer with $n \geq 2$, such that $(n-1)!+1 \equiv 0 \; (mod\; n)$, then show that n is prime. 5

(b) Show that if p is an odd prime, then $2(p-3)! \equiv -1 (mod\, p)$. 5

OPTION–B

Paper : MAT–HE–5026

(*Mechanics*)

1. Answer the following questions : ***(any ten)*** $1\times10=10$

(i) If a system of coplanar forces is in equilibrium, then what is the algebraic sum of the moment of the forces about any point in the plane ?

(ii) What is the resultant of the like parallel forces $P_1, P_2, P_3, ...$ acting on a body ?

(iii) If a particle moves under the action of a conservative system of forces, then what is the sum of its KE and PE ?

(iv) Define limiting equilibrium.

(v) Define the centre of gravity of a body.

(vi) Under what conditions the effect of a couple is not altered if it is transformed to a parallel plane ?

(vii) Write down the radial and cross-radial components of velocities of a particle moving on a plane curve at any point (r, θ) on it.

(viii) What is the resultant of a couple and a force in the same plane ?

(ix) What is dynamical friction ?

(x) What do you mean by terminal velocity ?

(xi) Define coefficient of friction.

(xii) What is the position of the point of action of the resultant of two equal like parallel forces acting on a rigid body ?

(xiii) What is the whole effect of a couple acting on a body ?

(xiv) Define simple harmonic motion.

(xv) What is the centre of gravity of a triangular lamina ?

(xvi) Define limiting friction.

(xvii) State the principle of conservation of energy.

(xviii) A particle moves on a straight line towards a fixed point O with an acceleration proportional to its distance from O. If x is the distance of the particle at time t from O, then write down its equation of motion.

2. Answer ***any five*** questions of the followoing : 2×5=10

(a) Write the laws of static friction.

(b) A particle moves in a circle of radius r with a speed v. Prove that its angular velocity is $\frac{v}{r}$.

(c) What are the general conditions of equilibrium of any system of coplanar forces ?

(d) The law of motion in a straight line is $s = \frac{1}{2}vt$. Prove that the acceleration is constant.

(e) Find the greatest and least resultant of two forces acting at a point whose magnitudes are P and Q respectively.

(f) Find the centre of gravity of an arc of a plane curve $y = f(x)$.

(g) State Hooke's law.

(h) Show that impulse of a force is equal to the momentum generated by the force in the given time.

(i) Write the expression for the component of velocity and acceleration along radial and cross radial direction in a motion of a particle in a plane curve.

(j) The speed v of a particle moving along x-axis is given by the relation $v^2 = n^2\left(8bx - x^2 - 12b^2\right)$. Prove that the motion is Simple Harmonic.

3. Answer ***any four*** questions of the following : 5×4=20

(a) The greatest and least resultants that two forces acting at a point can have magnitude P and Q respectively. Show that when they act at an angle α their resultant is $\sqrt{P^2 \cos^2 \frac{\alpha}{2} + Q^2 \sin^2 \frac{\alpha}{2}}$.

(b) I is the in centre of the triangle ABC. If three forces $\vec{P}, \vec{Q}, \vec{R}$ acting at I along $\overrightarrow{IA}, \overrightarrow{IB}, \overrightarrow{IC}$ are in equilibrium, prove that

$$\frac{P}{\sqrt{a(b+c-a)}} = \frac{Q}{\sqrt{b(c+a-b)}} = \frac{R}{\sqrt{c(a+b-c)}}$$

(c) Show that the resultant of three equal like parallel forces acting at the three vertices of a triangle passes through the centroid of the triangle.

(d) Prove that any system of coplanar forces acting on a rigid body can ultimately be reduced to a single force acting at any arbitrarily chosen point in the plane, together with a couple.

(e) Show that the sum of the Kinetic energy and Potential energy is constant throughout the motion when a particle of mass m falls from rest at a height h above ground.

(f) A point moves along a circle with constant speed. Find its angular velocity and acceleration about any point of the circle.

(g) Show that the work done against tension in stretching a light elastic string is equal to the product of its extension and the mean of the initial and final tension.

(h) A particle starts with velocity u and moves under retardation μ times of the distance. Show that the distance it travels before it comes to rest is $\frac{u}{\sqrt{\mu}}$.

4. Answer ***any four*** questions of the following : 10×4=40

(a) Forces P, Q and R act along the sides BC, CA and AB of a triangle ABC and forces P', Q' *and* R' act along OA, OB and OC, where O is the centre of the circumscribed circle, prove that

(i) $P\cos A + Q\cos B + R\cos C = 0$

(ii) $\dfrac{PP'}{a} + \dfrac{QQ'}{b} + \dfrac{RR'}{c} = 0$

(b) State and prove Lami's theorem. Forces P, Q and R acting along OA, OB and OC, where O is the circumcentre of triangle ABC, are in equilibrium. Show that

$$\frac{P}{a^2(b^2+c^2-a^2)} = \frac{Q}{b^2(c^2+a^2-b^2)} = \frac{R}{c^2(a^2+b^2-c^2)}$$

(c) *(i)* Find the centre of gravity of a uniform arc of the circle $x^2+y^2=a^2$ in the positive quadrant.

(ii) Find the centre of gravity of the arc of the asteroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ lying in the first quadrant.

(d) A particle moves in a straight line under an attraction towards a fixed point on the line varying inversely as the square of the distance from the fixed point. Investigate the motion.

(e) A particle moves in a straight line OA starting from the rest at A and moving with an acceleration which is directed towards O and varies as the distance from O. Discuss the motion of the particle. Hence define Simple Harmonic Motion and time period of the motion.

(f) Find the component of acceleration of a point moving in a plane curve along the initial line and the radius vector. Also find the component of acceleration perpendicular to initial line and perpendicular to radius vector.

(g) A particle is falling under gravity in a medium whose resistance varies as the velocity. Find the distance and velocity at any time t. Also find the terminal velocity of the particle.

(h) The velocity component of a particle along and perpendicular to the radius vector from λr and $\mu\theta$. Find the path and show that radial and transverse component of acceleration are

$$\lambda^2 r - \frac{\mu^2\theta^2}{r} \text{ and } \mu\theta\left(\lambda + \frac{\mu}{r}\right).$$

(i) Find the component of acceleration of a point moving in a plane curve along the initial line and the radius vector. Also find the component of acceleration perpendicular to initial line and perpendicular to radius vector.

(j) A particle moves in a straight line under an attaction towards a fixed point on the line varying inversely as the square of the distance from the fixed point. Investigate the motion.

OPTION–C

Paper : MAT–HE–5036

(Probability and Statistics)

1. Answer ***any ten*** questions from the following : $1 \times 10 = 10$

   (a) If A and B are mutually exclusive what will be the modified statement of

   $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

   (b) Define probability density funciton for a continuous random variable.

   (c) A random variable X can take all non-negative integral values, and the probability that X takes the value r is

   $P(X = r) = A\alpha^r \, (0 < \alpha < 1)$. Find $P(X = 0)$.

   (d) If X and Y are two random variables and $var\ (X - Y) \neq var(X) - var(Y)$ then what is the relation between X and Y.

   (e) Test the velocity of the following probability distribution :

x	-1	0	1
$P(x)$	0.4	0.4	0.3

(f) Define Negative Binomial distribution for a random variable X with parameter r.

(g) What are the relations between mean, median and mode of a normal distribution ?

(h) Write the equation of the line of regression of x on y.

(i) What is the variance of the mean of a random sample ?

(j) Define moment generating function of a random variable X about origin.

(k) What are the limits for correlation coefficents ?

(l) For a Bernoulli random variable X with $P(X=0)=1-P$ and $P(X=1)=P$ write $E(X)$ and $V(X)$ in terms of P.

(m) If X is a random variable with mean μ and variance σ^2, then for any positive number k, find Chebychev's inequality.

(n) A continuous random variable X follow the probability law $f(x)=Ax^2, 0\le x\le 1$. Determine A.

(o) If X and Y are two random variables then find $cov(x, y)$.

(p) If a is constant then find $E(a)$ and $var(a)$.

(q) If X and Y are two independent. Poisson variates, then XY is a _____ variate. *(Fill in the blank)*

(r) If a non-negative real valued function f is the probability density function of some continuous random variable,then what is the value of $\int_{-\alpha}^{\alpha} f(x)\,dx$?

2. Answer the following questions : ***(any five)*** $2\times5=10$

(a) If A and B are independent events, then show that A and B are also independent.

(b) If X have the p.m.f

$$f(x)=\frac{x}{10}, x=1,2,3,4$$

then find $E(X^2)$

(c) With usual notation for a binomial variate X, given that

$9\,P(x=4)=P(x=2)$ when $n=6$.

Find the value of p and q.

(d) If X is a continuous random variable whose probability density function is given by

$$f(x)=\begin{cases}\frac{3}{8}\left(4x-2x^2\right) & 0<x<2\\ 0, & \text{otherwise}\end{cases}$$

then find $P\{X>1\}$.

(e) Show that for a normal standard variate z, $E(z)=0$ and $V(z)=1$.

(f) The number of items produced in a factory during a week is a random variable with mean 50. If the variance of a week's production is 25, then what is the probability that this week's production will be between 40 and 60.

(g) Define probability mass function and probability density function for a random varibale X.

(h) If X is a random Poisson variate with parameter λ, then show that

$$P(X \geq n) - P(X \geq n+1) = \frac{e^{-\lambda} m^n}{\lfloor n}$$

(i) If $M_x(t)$ is a moment generating function of a random variable X with parameter t then show that $M_{cX}(t) = M_X(ct)$, c is a constant.

(j) If X and Y are independent random variables with characteristic functions $\varphi_X(w)$ and $\varphi_Y(w)$ respectively then show that

$$\varphi_{X+Y}(w) = \varphi_X(w)\varphi_Y(w)$$

3. Answer ***any four*** questions from the following : 5×4=20

(a) If X is a discrete random variable having probability mass function 2+2+1=5

mass point	0	1	2	3	4	5	6	7
$p(X=x)$	0	k	$2k$	$3k$	$4k$	k^2	$2k^2$	$7k^2+k$

Determine :

(i) k

(ii) $p(X<6)$ and

(iii) $p(X \geq 6)$

(b) If X and Y are two independent random variables then show that

$$var(X+Y) = var(X) + var(Y)$$

(c) If the probability that an individual will suffer a bad reaction from injective of a given serum is 0.001 determine the probability that out of 2000 individuals

(i) exactly 3,

(ii) more than 2 individual

will suffer a bad reaction. 2+3=5

(d) Two random variables X and Y are jointly distributed as follows :

$$f(x,y) = \frac{2}{\pi}\left(1 - x^2 - y^2\right), 0 < x^2 + y^2 < 1$$

Find the marginal distribution of X.

(e) State and prove weak law of large numbers.

(f) If X and Y are independent random variables having common density function

$$f(x) = e^{-x}, x > 0$$
$$0, \text{ otherwise}$$

Find the density function of the random variable X/Y.

(g) If X and Y are independent Poisson variates such that

$P(x=1)=P(x=2)$ and

$P(y=2)=P(y=3)$

Find the variance of $x-2y$.

(h) Prove that regression coefficients are independent of the change of origin but not of scale.

4. Answer ***any four*** from the following questions : 10×4=40

(a) (i) What is meant by partition of a sample space S? If $Hi\,(i=1,2,..n)$ is a partition of the sample space S, then for any event A, prove that

$$P(Hi/A)=\frac{P(Hi)P(A/Hi)}{\sum_{i=1}^{n}P(Hi)P(A/Hi)}$$ 5

(ii) If X is a random variable with the following probability distribution :

x:	-3	6	9
$P(X=x)$:	1/6	1/2	1/3

Find $E(X), E(X^2)$ and $var(X)$ 5

(b) Two random variables X and Y have the following joint probability density function : 2+2+3+3=10

$$f(x,y)=\begin{cases}2-x-y, & 0\le x\le 1,\ 0\le y\le 1\\ 0, & \text{otherwise}\end{cases}$$

Find (i) marginal probability density function of X and Y (ii) conditional density function (iii) var (X) and var (Y) (iv) co-variance between X and Y.

(c) (i) Let X be a random variable with mean μ and variance r^2. Show that $E(x-b)^2$ as a function of b is minimum when $b=\mu$. 5

(ii) A bag contains 5 balls and it is known how many of these are white. Two balls are drawn and are found to be white. What is the probability that all are white ? 5

(d) (i) If X is a random variable then prove that 5

$$var(X)=var[E(X/Y)]+E[var(X/Y)]$$

(ii) Find the probability that in a family of 4 children there will be *(a) at least one* boy *(b) at least one* boy and *at least one* girl. 5

(e) What are the chief characteristics of the normal distribution and normal curve ?

(f) (i) Show that mean and variance of a Poisson distribution are equal. 5

(ii) Determine the binomial distribution for which the mean is 4 and variance is 3 and find its mode. 5

(g) (i) Prove that independent variables are uncorrelated. With the help of an example show that the converse is not true. 5

(ii) Find the angle between the two lines of regression 5

$$y - \overline{y} = \frac{r\sigma_y}{\sigma_x}(x - \overline{x})$$

and $x - \overline{x} = r\dfrac{\sigma_x}{\sigma_y}(y - \overline{y})$

(h) (i) A function $f(x)$ of x is defined as follows :

$$f(x) = 0 \text{ for } x < 2$$
$$= \frac{1}{18}(3+2x) \text{ for } 2 \le x \le 4$$
$$= 0, \text{ for } x > 4$$

Show that it is a density function. Also find the probability that a variate with this density will lie in the interval $2 \le x \le 3$. 5

(ii) A random variable X can assume values 1 and – 1 with probability $\frac{1}{2}$ each. Find

(i) moment generating function,

(ii) characteristics function. 5

(i) (i) Find the median of a normal distribution. 5

(ii) A random variable X has density function given by 5

$$f(x) = \begin{cases} 2e^{-2x}, x \ge 0 \\ 0, \quad x < 0 \end{cases}$$

Find (i) mean with the help of m.g.f. (ii) $P[|x-\mu|>1]$.

(j) (i) The diameter say x, of an electric cable is assumed to be continuous random variable with p.d.f

$$f(x) = 6x(1-x),\ 0 \le x \le 1$$

(a) Check that the above is a p.d.f,

(b) Determine the value of k such that $P(X < K) = P(X > K)$ 5

(ii) If 3% of electric bulbs manufactured by a company are defective, using Poisson's distribution find the probability that in a sample of 100 bulbs exactly 5 bulbs are defective 5

$$\left[\text{Given } e^{-3} = 0.04979\right]$$
