3 (Sem–4/CBCS) MAT HC 3

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–4036

(Ring Theory)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×10=10

(a) Give an example of an infinite noncommutative ring that does not have a unity.

(b) Define an integral domain.

(c) What is the characteristic of the ring of 2×2 matrices over integers?

(d) In an integral domain, if $a \neq 0$ and $ab = ac$, then prove that $b = c$.

Contd.

(e) Show that $2Z \cup 3Z$ is not a subring of Z.

(f) Prove that the correspondence $x \to 5x$ from Z_5 to Z_{10} does not preserve addition.

(g) Characteristic of every field is

(i) 0

(ii) an integer

(iii) either 0 or prime

(iv) either 0 or not prime

(Choose the correct option)

(h) Which of the following is not an integral domain ?

(i) $Z[x]$

(ii) $\{a + b\sqrt{2} : a, b \in Z\}$

(iii) Z_3

(iv) Z_6

(Choose the correct option)

(c) Prove that the ring of integers Z is a principal ideal domain.

(d) Let ϕ be a homomorphism from a ring R to a ring S. If A is a subring of R and B is an ideal of S, prove that

(i) $\phi(A) = \{\phi(a) \mid a \in A\}$ is a subring of S.

(ii) $\phi^{-1}(B) = \{x \in R \mid \phi(x) \in B\}$ is an ideal of R. $2\frac{1}{2}+2\frac{1}{2}=5$

(e) Let F be a field, $a \in F$ and $f(x) \in F[x]$. Prove that a is a zero of $f(x)$ if and only if $x-a$ is a factor of $f(x)$.

(f) Let F be a field, I a nonzero ideal in $F[x]$, and $g(x)$ an element of $F(x)$. Show that $I = \langle g(x) \rangle$ if and only if $g(x)$ is a nonzero polynomial of minimum degree in I.

Answer **either** *(a) and (b)* **or** *(c) and (d) of the following questions :* $10\times4=40$

4. (a) Prove that a finite integral domain is a field. Hence show that for every prime p, Z_p, the ring of integers modulo p, is a field. $4+2=6$

(b) Prove that the only ideals of a field are $\{0\}$ and F itself.

(c) Show that the ring of integers is an Euclidean domain.

(d) If R is a commutative ring with unity and A is an ideal of R, show that R/A is a commutative ring with unity.

(e) Let $f(x) = x^3 + 2x + 4$ and $g(x) = 3x + 2$ in $Z_5[x]$. Determine the quotient and remainder upon dividing $f(x)$ by $g(x)$.

3. Answer ***any four*** questions of the following : 5×4=20

(a) Prove that

$$Z[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in Z\}$$

is a ring under the ordinary addition and multiplication of real numbers.

(b) (i) If I is an ideal of a ring R such that 1 belongs to I, then show that $I = R$.

(ii) Let R be a ring and $a \in R$. Show that $S = \{ r \in R \mid ra = 0 \}$ is an ideal of R. 2+3=5

(c) Prove that the ring of integers Z is a principal ideal domain.

(d) Let ϕ be a homomorphism from a ring R to a ring S. If A is a subring of R and B is an ideal of S, prove that

(i) $\phi(A)=\{\phi(a)\,|\,a\in A\}$ is a subring of S.

(ii) $\phi^{-1}(B)=\{x\in R\,|\,\phi(x)\in B\}$ is an ideal of R. 2½+2½=5

(e) Let F be a field, $a\in F$ and $f(x)\in F[x]$. Prove that a is a zero of $f(x)$ if and only if $x-a$ is a factor of $f(x)$.

(f) Let F be a field, I a nonzero ideal in $F[x]$, and $g(x)$ an element of $F(x)$. Show that $I=\langle g(x)\rangle$ if and only if $g(x)$ is a nonzero polynomial of minimum degree in I.

Answer ***either*** *(a) and (b)* ***or*** *(c) and (d) of the following questions :* 10×4=40

4. (a) Prove that a finite integral domain is a field. Hence show that for every prime p, Z_p, the ring of integers modulo p, is a field. 4+2=6

(b) Show that $\frac{R[x]}{\langle x^2+1 \rangle}$ is a field. 4

OR

(c) Prove that every field is an integral domain. Is the converse true? Justify with an example. 2+1=3

(d) Define prime ideal and maximal ideal of a ring. Show that $\langle x \rangle$ is a prime ideal of $Z[x]$ but not a maximal ideal of it. 2+5=7

5. (a) Let ϕ be a homomorphism from a ring R to a ring S. Prove that ϕ is an isomorphism if and only if ϕ is onto and $ker\phi = \{r \in R \mid \phi(r) = 0\} = \{0\}$. 5

(b) If ϕ is an isomorphism from a ring R to a ring S, then show that ϕ^{-1} is an isomorphism from S to R. 5

OR

(c) Let R be a ring with unity e. Show that the mapping $\phi : \mathbb{Z} \to R$ given by $n \to ne$ is a ring homomorphism. 5

(d) Define kernel of a ring homomorphism. Let ϕ be a homomorphism from a ring R to a ring S. Prove that $ker\phi$ is an ideal at R. 1+4=5

6. (a) State and prove the second isomorphism theorem for rings. 1+7=8

(b) Let R be a commutative ring of characteristic 2. Show that the mapping $a \to a^2$ is a ring homomorphism from R to R. 2

OR

(c) State and prove the third isomorphism theorem for rings. 1+6=7

(d) Prove that every ideal of a ring R is the kernel of a ring homomorphism of R. 3

7. (a) Let F be a field. If $f(x) \in F[x]$ and $deg\, f(x) = 2$ or 3, then prove that $f(x)$ is reducible over F if and only if $f(x)$ has a zero in F. 4

(b) In a principal ideal domain prove that an element is an irreducible if and only if it is a prime. 6

OR

(c) Let p be a prime and suppose that $f(x) \in Z[x]$ with $deg\, f(x) \geq 1$. Let $\overline{f(x)}$ be the polynomial in $Z_p[x]$ obtained from $f(x)$ by reducing all the coefficients of $f(x)$ modulo p. If $\overline{f(x)}$ is irreducible over Z_p and $deg\, \overline{f(x)} = deg\, f(x)$, then prove that $f(x)$ is irreducible over Q. 5

(d) Let

$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0 \in Z[x]$.

If there is a prime p such that

$p \nmid a_n$, $p | a_{n-1}, ..., p | a_0$ and $p^2 \nmid a_0$,

then prove that $f(x)$ is irreducible over Q. 5
