

Total number of printed pages-28

3 (Sem-6/CBCS) MAT HE 1/2/3/4

2023

**MATHEMATICS**

(Honours Elective)

**Answer the Questions from any one Option.**

**OPTION - A**

**(Boolean Algebra and Automata Theory)**

Paper : MAT-HE-6016

Full Marks : 80

Time : Three hours

**OPTION - B**

**(Biomathematics)**

Paper : MAT-HE-6026

Full Marks : 80

Time : Three hours

**OPTION - C**

**(Mathematical Modeling)**

Paper : MAT-HE-6036

Full Marks : 60

Time : Three hours

**OPTION - D**

**(Hydromechanics)**

Paper : MAT-HE-6046

Full Marks : 80

Time : Three hours

**The figures in the margin indicate full marks for the questions.**

Contd.

## OPTION-A

### (Boolean Algebra and Automata Theory)

Paper : MAT-HE-6016

1. Answer the following questions :  $1 \times 10 = 10$

(a) A relation  $\leq$  on a set  $P$  is called quasi-order, if

(i) reflexive, transitive and antisymmetric

(ii) reflexive and antisymmetric

(iii) transitive and antisymmetric

(iv) None of the above

(Choose the correct answer)

(b) An ordered set  $P$  is an antichain if \_\_\_\_\_ in  $P$  only if \_\_\_\_\_.

(Fill in the blanks)

(c) Let  $P^D$  be the dual of any ordered set  $P$ . Then

(i)  $x \leq y$  holds in  $P^D$  if  $x \leq y$  holds in  $P$

(ii)  $x \leq y$  holds in  $P^D$  if  $y \leq x$  holds in  $P$

(iii)  $x \leq y$  holds in  $P^D$  if  $x = y$  holds in  $P$

(iv) None of the above

(Choose the correct answer)



(d) Define lattice homomorphism.

(e) Let  $L$  be a lattice and  $a, b \in L$ . If  $a \leq b$ , then

(i)  $a \vee b = b, a \wedge b = a$

(ii)  $a \vee b = b$  but not  $a \wedge b = a$

(iii)  $a \wedge b = a$  but not  $a \vee b = b$

(iv) None of the above

*(Choose the correct answer)*

(f) Define conjunctive normal form.

(g) For all  $x, y$  in a Boolean algebra,

(i)  $(x \wedge y)' = x' \vee y'$  and  $(x \vee y)' = x' \wedge y'$

(ii)  $(x \wedge y)' = x' \wedge y'$  and  $(x \vee y)' = x' \vee y'$

(iii)  $(x \wedge y)' = y'$  and  $(x \vee y)' = x'$

(iv) None of the above

*(Choose the correct answer)*

(h) Define Boolean polynomial function.

(i) What is the empty string?

(j) Define closure properties of regular languages.

2. Answer the following questions :  $2 \times 5 = 10$

(a) Prove that the elements of any arbitrary lattice satisfy the following inequalities :

$$(i) \quad x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$$

$$(ii) \quad x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$$

(b) Prove that every chain is a distributive lattice.

(c) Define NFA.

(d) Define atom. Prove that every atom of a lattice with zero is join-irreducible.

(e) Prove that if  $L$  and  $M$  are regular languages, then  $L \cup M$  is also a regular language.

3. Answer **any four** questions from the following :  $5 \times 4 = 20$

(a) (i) Prove that two finite ordered set  $P$  and  $Q$  are order-isomorphic if and only if they can be drawn with identical diagrams.



- (ii) Define monomorphism. Let  $f$  be a monomorphism from the lattice  $L$  into the lattice  $M$ . Show that  $L$  is isomorphic to a sublattice  $M$ .
- (b) (i) Let  $C_1$  and  $C_2$  be the finite chains  $\{0, 1, 2\}$  and  $\{0, 1\}$  respectively. Draw the Hasse diagram of the product lattice  $C_1 \times C_2 \times C_3$ .
- (ii) Let  $L$  be a distributive lattice with 0 and 1. Prove that if  $a$  has a complement  $a'$ , then  $a \vee (a' \wedge b) = a \vee b$ .
- (c) (i) State and prove De Morgan's laws of a Boolean algebra.
- (ii) Let  $f: B_1 \rightarrow B_2$  be a Boolean homomorphism. Then prove the following :
- (1)  $f(0) = 0, f(1) = 1$
- (2) For all  $x, y \in B_1$   
 $x \leq y \Rightarrow f(x) \leq f(y)$ .
- (d) Let  $p, q \in P_n; p \sim q$  and let  $B$  be an arbitrary Boolean algebra. Then, prove that  $\bar{p}_B = \bar{q}_B$ .

- (e) Prove that a language  $L$  is accepted by some DFA if and only if  $L$  is accepted by some NFA.
- (f) Prove that every regular language is a context-free language.

4. Answer the following questions :  $10 \times 4 = 40$

(a) (i) Let  $P$  and  $Q$  be finite ordered sets and let  $f : P \rightarrow Q$  be a bijective map. Then, prove that the following are equivalent :

(1)  $f$  is an order-isomorphism;

(2)  $x < y$  in  $P$  if and only if  $f(x) < f(y)$  in  $Q$ ;

(3)  $x < y$  in  $P$  if and only if  $f(x) < f(y)$  in  $Q$ . 5

(ii) Let  $P$  be an ordered set. Then, prove that

$$O(P \oplus 1) \cong O(P) \oplus 1 \text{ and}$$

$$O(1 \oplus P) \cong \oplus 1 O(P) \quad 5$$



**OR**

Let  $P$  be a finite ordered set.

(i) Show that  $Q = \downarrow \text{Max } Q$ , for all  $Q \in O(P)$

(ii) Establish a one-to-one correspondence between the elements of  $O(P)$  and antichains in  $P$

(iii) Hence show that for all  $x \in P$ ,

$$|O(P)| = |O(P \setminus \{x\})| + |O(P \setminus (\downarrow x \cup \uparrow x))|$$

10

(b) (i) Let  $L$  be a distributive lattice and let  $p \in L$  be join-irreducible with  $p \leq a \vee b$ . Then, prove that  $p \leq a$  or  $p \leq b$ . 5

(ii) Prove that generalized distributive inequality for lattices

$$y \wedge \left( \bigvee_{i=1}^n x_i \right) \geq \bigvee_{i=1}^n (y \wedge x_i).$$

5

**OR**

(iii) Let  $B$  be a Boolean algebra. Then, prove that the set  $P_n(B)$  is a Boolean algebra and subalgebra of the Boolean algebra  $F_n(B)$  of all functions from  $B_n$  into  $B$ . 5

(iv) Find the DNF of

$$x_1(x_2 + x_3)' + (x_1x_2 + x_3')x_1 \quad 5$$

(c) (i) Prove that a polynomial  $p \in P_n$  is equivalent to the sum of all prime implications of  $p$ . 5

(ii) Find three prime implications of  $xy + xy'z + x'y'z$ . 5

**OR**

(iii) Determine the symbolic representation of the circuit given by

$$p = (x_1 + x_2 + x_3)(x_1' + x_2')(x_1x_3 + x_1'x_2)(x_2' + x_3)$$

5



(iv) Design a switching circuit that enables you to operate one lamp in a room from four different switches in that room. 5

(d) (i) If  $L$ ,  $M$  and  $N$  are any languages, then prove that

$$L(M \cup N) = LM \cup LN. \quad 5$$

(ii) If  $L$  is a regular language over alphabet  $\Sigma$ , then  $\bar{L} = \Sigma^* - L$  is also a regular language. 5

**OR**

(iii) Consider the CFG  $K$  defined by productions

$$S \rightarrow aSbS \mid bSaS \mid \varepsilon$$

Prove that  $L(K)$  is the set of all strings with an equal number of  $a$ 's and  $b$ 's. 5

(iv) Let  $G = (V, T, P, S)$  be a CFG, and suppose that there is a derivation

$$A \xrightarrow[G]{*} w, \text{ where } w \text{ is in } T^*. \text{ Then,}$$

prove that the recursive inference procedure applied to  $G$  determines that  $w$  is in the language of variable  $A$ . 5

**OPTION-B**

**(Biomathematics)**

Paper : MAT-HE-6026

1. Answer the following questions :  $1 \times 10 = 10$

- (a) What is an autonomous system ?
- (b) The zero equilibrium/positive equilibrium is often not a desired state in biological system.  
*(Choose the correct answer)*
- (c) Write a difference between continuous growth and discrete growth.
- (d) Give an example of nonlinear, autonomous second order difference equation.
- (e) Write *one* use of Routh-Hurwitz criteria.
- (f) Equilibria are also known as
- (a) steady state
  - (b) fixed points
  - (c) critical points
  - (d) All of the above
- (Choose the correct answer)*



- (g) Write the condition that a first order partial derivative of a system is locally asymptotically stable.
- (h) Write the condition that the equilibrium  $\bar{x}$  of  $\frac{dx}{dt} = f(x)$  is hyperbolic.
- (i) Write the three population classes in Kermack-McKendrick model.
- (j) Define a characteristic polynomial for second order equation.

2. Answer the following questions :  $2 \times 5 = 10$

- (a) Define a difference equation of order  $k$ .
- (b) State Frobenius theorem.
- (c) Distinguish between local stability and global stability.
- (d) Consider the linear differential equation

$$\frac{d^3 x}{dt^3} + 4 \frac{d^2 x}{dt^2} + \frac{dx}{dt} + ax = 0$$

Show that its solution approaches zero.

(e) For the linear differential equation

$$\frac{dx}{dt} = AX, \text{ the matrix } A \text{ is given by}$$

$$A = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}. \text{ Find the eigenvalues.}$$

3. Answer **any four** questions :  $5 \times 4 = 20$

(a) The difference equation is given by

$$x_{t+4} + ax_t = 0.$$

Find its characteristic equation and its solutions.

(b) Find the eigenvalues and eigenvectors of matrix  $A$  when

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Then find the general solutions to

$$x(t+1) = Ax(t).$$

(c) Find all the equilibria for the difference equation  $x_{t+1} = ax_t \exp(-rx_t)$ ,  $a, r > 0$ .



(d) Consider the differential equation

$$x'''(t) - 4x''(t) = 0$$

where  $x'' = \frac{d^2x}{dt^2}$  and so on.

Find its characteristic equation and its roots or eigenvalues and verify that the solutions are linearly independent or not.

(e) A mathematical model for the growth of a population is

$$\frac{dx}{dt} = \frac{2x^2}{1+x^4} - x = f(x), \quad x(0) \geq 0$$

where  $x$  is the population density. Find the equilibria and determine their stability.

(f) Suppose an SIS epidemic model with disease-related deaths and a growing population satisfies

$$\frac{dN}{dt} = N(b - cN) - \alpha I, \quad b, c, \alpha > 0$$

(i) Find the differential equations satisfied by the proportions

$$i(t) = \frac{I(t)}{N(t)} \quad \text{and} \quad s(t) = \frac{S(t)}{N(t)}$$

Then find the basic reproduction number.

(ii) Do the dynamics of  $N(t)$  change with disease? Is it possible for  $N(t) \rightarrow 0$ ? Note that  $m(N) = CN$

$$\text{and } \frac{dN}{dt} = N(b - CN - ai).$$

4. Answer the following questions :  $10 \times 4 = 40$

(a) Find the general solution to the non-homogeneous linear difference equation

$$x_{t+2} + x_{t+1} = 6x_t = 5$$

**Or**

Suppose the Leslie matrix is given by

$$L = \begin{pmatrix} 0 & \frac{3a^2}{2} & \frac{3a^3}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}, \quad a > 0$$

(i) Find the characteristic equation, eigenvalues and inherent net reproduction number  $R_0$  of  $L$ .



- (ii) Show that  $L$  is primitive.
- (iii) Find the stable age distribution.
- (b) The following epidemic model is referred to as an SIS epidemic model. Infected individuals recover but do not become immune. They become immediately susceptible again.

$$S_{t+1} = S_t - \frac{\beta}{N} I_t S_t + (\gamma + b) I_t$$

$$I_{t+1} = I_t (1 - \gamma - b) + \frac{\beta}{N} I_t S_t$$

Assume that  $0 < \beta < 1$ ,  $0 < b + \gamma < 1$

$S_0 + I_0 = N$  and  $S_0, I_0 > 0$

- (i) Show that  $S_t + I_t = N$   
for  $t = 1, 2, \dots$
- (ii) Show that there exist two equilibria and they are both non-negative if  $R_0 = \frac{\beta}{b + \gamma} \geq 1$ .

**Or**

Discuss a predator-prey model with a suitable example by finding its equilibria, local stability and global stability.

- (c) State briefly a measles model with vaccination.

**Or**

Show that the solution to the pharmacokinetics model is

$$x(t) = \frac{1}{a} (1 - e^{-at})$$

$$y(t) = \frac{1}{b} + \frac{e^{-at}}{a-b} - \frac{ae^{-bt}}{b(a-b)}$$

- (d) For the following differential equation, find the equilibria, then graph the phaseline diagram. Use the phaseline diagram to determine the stability of equilibrium

$$\frac{dx}{dt} = x(a-x)(x-b)^2, \quad 0 < a < b.$$

**Or**

Discuss briefly about simple Kermack-McKendric epidemic model.



**OPTION-C**

**(Mathematical Modeling)**

Paper : MAT-HE-6036

1. Answer the following questions :  $1 \times 7 = 7$

- (a) Write Legendre's equation of order  $n$ .
- (b) When does a power series converge if  $f$  be the radius of convergence and  $0 < \rho < \infty$  ?
- (c) Write the value of  $\Gamma 3$ .
- (d) Find the Laplace transform of  $F(t) = 1$ .
- (e) Monte Carlo simulation is a probabilistic/logistic model.  
(Choose the correct answer)
- (f) The linear congruence method was introduced by \_\_\_\_\_ .  
(Fill in the blank)

(g) Which one is not a high level simulation language ?

(i) GPSS

(ii) SPSS

(iii) SIMAN

(iv) DYNAMO

*(Choose the correct answer)*

2. Answer the following questions :  $2 \times 4 = 8$

(a) Show that  $\sqrt{x+1} = x\sqrt{x}$ .

(b) Find the inverse Laplace transform of

$$F(s) = \frac{1}{s(s-3)}$$

(c) Write *two* advantages of Monte Carlo simulation.

(d) Why is sensitivity analysis important in linear programming ?



3. Answer **any three** questions of the following : 5×3=15

(a) Solve the equation

$$y' + 2y = 0$$

(b) Find the exponents in the possible Frobenius series solutions of the equation

$$2x^2(1+x)y'' + 3x(1+x)^3y' - (1-x^2)y = 0$$

(c) Suppose that  $m$  is a positive integer. Show that

$$\Gamma\left(m + \frac{2}{3}\right) = \frac{2.5.8\dots(3m-1)}{3^m} \sqrt{\frac{2}{3}}$$

(d) Solve the equation

$$4x^2y'' + 8xy' + (x^4 - 3)y = 0$$

(e) Write briefly about different steps of the simplex method.

4. Answer the following : 10×3=30

(a) Solve the initial value problem

$$(t^2 - 2t - 3) \frac{d^2y}{dt^2} + 3(t-1) \frac{dy}{dt} + y = 0;$$

$$y(1) = 4, \quad y'(1) = -1$$

**Or**

Find the Frobenius series solutions of  
 $xy'' + 2y' + xy = 0$ .

(b) Using Monte Carlo simulation, write an algorithm to calculate an approximation to  $\pi$  by considering the number of random points selected inside the quarter circle.

$$Q : x^2 + y^2 = 1, \quad x \geq 0, \quad y \geq 0$$

where the quarter circle is taken to be inside the square

$$S : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1.$$

**Or**

Solve the equation  $y'' + y = 0$ .

(c) Write briefly about middle square method.



Or

A small harbor has unloading facilities for ships. Only one ship can be unloaded at any time. The unloading time required for a ship depends on the type and amount of cargo and varies from 45 to 90 minutes.

Below is given a situation with 5 ships :

	Ship 1	Ship 2	Ship 3	Ship 4	Ship 5
Time between successive ships : (in minutes)	30	15	20	25	120
Unloading time :	40	35	60	45	75

- (i) Draw the time line diagram depicting clearly the situation for each ship, the idle time for the harbor and the waiting time.
- (ii) List the waiting time for all the ships and find the average waiting time.
-

## OPTION-D

### (Hydromechanics)

Paper : MAT-HE-6046

1. Answer the following questions :  $1 \times 10 = 10$

- (a) What happens when there is an increase of pressure at any point of a liquid at rest under given external forces?
- (b) State Charles' law.
- (c) What is internal energy?
- (d) Define adiabatic expansion.
- (e) Give an example of application of atmospheric pressure in daily life.
- (f) Define ideal fluid.
- (g) Potential flow is the ..... flow of an inviscid or perfect flow.

*(Fill in the gap)*



(h) Equation of continuity by Euler's method is

(i)  $\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{a} = 0$

(ii)  $\frac{\partial \rho}{\partial t} - \rho \nabla \cdot \vec{a} = 0$

(iii)  $\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho \vec{a}) = 0$

(iv) None of the above

*(Choose the correct option)*

(i) Streamlines and pathlines become the same when the motion is .....

*(Fill in the gap)*

(j) Velocity potential  $\phi$  satisfies which of the following equations ?

(i) Bernoulli

(ii) Cauchy

(iii) Laplace

(iv) None of the above

*(Choose the correct option)*

2. Answer the following questions :  $2 \times 5 = 10$

(a) Show that the surfaces of equal pressure are intersected orthogonally by the lines of force.

(b) Define field of force and line of force with examples.

(c) If  $\rho_0$  and  $\rho$  be the densities of a gas at  $0^\circ$  and  $t^\circ$  Centigrade respectively, then establish the relation  $\rho_0 = \rho(1 + \alpha t)$

$$\text{where } \alpha = \frac{1}{273}.$$

(d) Distinguish between the streamlines and pathlines.

(e) Give examples of irrotational and rotational flows.

3. Answer the following questions : **(any four)**

$$5 \times 4 = 20$$

(a) Determine the necessary condition that must be satisfied by a given distribution of forces  $X$ ,  $Y$ ,  $Z$ , so that the fluid may maintain equilibrium.



- (b) A quadrant of a circle is just immersed vertically with one edge in the surface, in a liquid, the density of which varies as the depth. Determine the centre of pressure (C.P.).
- (c) A box is filled with a heavy gas at a uniform temperature. Prove that if  $a$  is the altitude of the highest point above the lowest and  $p$  and  $p'$  are the pressures at these two points, the ratio of the pressure to the density at any point is equal to

$$\frac{ag}{\log p'/p}$$

- (d) If  $w$  is the area of cross-section of a stream filament, prove that the equation of continuity is

$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial s}(\rho w q) = 0$$

where  $\delta s$  is an element of arc of the filament in the direction of flow and  $q$  is the speed.

(e) Determine the acceleration of a fluid particle when velocity distribution is  $\vec{a} = \hat{i}(Ax^2yt) + \hat{j}(By^2zt) + \hat{k}(Czt^2)$  where  $A, B, C$  are constants. Also find the velocity components.

(f) The velocity field at a point in fluid is given by  $\vec{a} = (x/t, y, 0)$ . Obtain the pathlines.

4. Answer the following questions :  $10 \times 4 = 40$

(a) A mass of homogeneous liquid contained in a vessel revolves uniformly about a vertical axis. You are required to determine the pressure at any point and the surfaces of equal pressure.

**OR**

A mass  $m$  of elastic fluid is rotating about an axis with uniform angular velocity  $\omega$ , and is acted on by an attraction towards a point in that axis equal to  $\mu$  times the distance,  $\mu$  being greater than  $\omega^2$ . Prove that the equation of a surface of equal density  $\rho$  is

$$\mu(x^2 + y^2 + z^2) - \omega^2(x^2 + y^2) = k \log \left\{ \frac{\mu(\mu - \omega^2)^2}{8\pi^3} \cdot \frac{m^2}{\rho^2 k^3} \right\}$$



- (b) A hemispherical bowl is filled with water and two vertical planes are drawn through its central radius, cutting off a semi-lune of the surface. If  $2\alpha$  be the angle between the planes, prove that the angle which the resultant pressure on the surface makes with the vertical

$$= \tan^{-1}\left(\frac{\sin \alpha}{\alpha}\right).$$

**OR**

A gaseous atmosphere in equilibrium is such that  $p = k\rho^\gamma = R\rho T$  where  $p, \rho, T$  are the pressure, density and temperature and  $k, \gamma, R$  are constants. Prove that the temperature decreases upwards at a constant rate  $\alpha$ , so

that  $\frac{dT}{dZ} = -\alpha = -\frac{g}{R} \cdot \frac{\gamma-1}{\gamma}$ . In a certain

atmosphere of uniform composition  $T = T_0 - \beta z$  where  $T_0$  and  $\beta$  are constants and  $\beta < \alpha$ . Find the pressure and density and show that they both

vanish at height  $\frac{T_0}{\beta}$ .

- (c) Derive the equation of continuity in Cartesian coordinates. Also what happen, if the fluid is homogeneous and incompressible.

**OR**

Derive the equation of continuity by the Lagrangian method.

- (d) The velocity components for a two-dimensional fluid system can be given in Eulerian system by

$$U = 2x + 2y + 3t$$

$$V = x + y + \frac{t}{2}$$

Find the displacement of a fluid particle in the Lagrangian system.

**OR**

Obtain Euler's equation of motion of a non-viscous fluid in the form

$$\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho} \nabla P.$$