Total number of printed pages–20

3 (Sem–6/CBCS) MAT HE 5/6/7

2023

MATHEMATICS

(Honours Elective)

Answer the Questions from any one Option.

OPTION–E

(Rigid Dynamics)

Paper : MAT–HE–6056

OPTION–F

(Group Theory–II)

Paper : MAT–HE–6066

OPTION–G

(Mathematical Finance)

Paper : MAT–HE–6076

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

Contd.

OPTION–E

(Rigid Dynamics)

Paper : MAT–HE–6056

1. Answer the following questions : 1×10=10

   (a) The distance between *any two* particles of a rigid body always varies.
   (State True ***or*** *False)*

   (b) Define product of inertia of a body.

   (c) When are *two* systems said to be equimomental ?

   (d) Define the momental ellipsoid.

   (e) Define centroid of a system of particles.

   (f) Write the principle of conservation of linear momentum under finite forces.

   (g) State the parallel axes theorem.

   (h) What are generalized co-ordinates ?

   (i) Define angular momentum of a system of particles.

   (j) Define centre of oscillation.

2. Answer the following questions : 2×5=10

(a) Write a very short note on simple equivalent pendulum.

(b) Write the necessary and sufficient conditions for two systems to be equimomental.

(c) Obtain the scaler equations of motion of a particle of mass M, placed at the centre of inertia of the body and acted on by the forces $\Sigma X, \Sigma Y, \Sigma Z$ parallel and equal to the external forces acting on different points of the body.

(d) A rigid body consists of three particles of masses 2 units, 3 units and 5 units located at the points (1, 1, 0), (–1, 0, 1) and (0, 1, 1) respectively. Find the products of inertia about

(i) x axis and y axis;

(ii) y axis and z axis.

(e) A rigid body of mass 5 units rotates with angular velocity $\vec{\omega} = (1, 1, -1)$ and has the angular momentum $\vec{\Omega} = (4, 2, 1)$. Find the kinetic energy of the body.

3. Answer ***any four*** questions : 5×4=20

(a) Show that the momental ellipsoid at the centre of an ellipsoid is $(b^2+c^2)x^2+(a^2+c^2)y^2+(a^2+b^2)z^2=$ constant.

(b) If α, β, γ and h be the distances of the vertices and the centre of inertia of a uniform triangular lamina of mass m from any line, prove that the moment of inertia about that line is $\frac{1}{12}m\left[(\alpha^2+\beta^2+\gamma^2)+9h^2\right]$

(c) Derive the general vector equations of motion of a rigid body mentioning the name of the principle used in the derivation.

(d) Find the time of complete oscillation of a compound pendulum consisting of a rod of mass m and length a carrying at one end a sphere of mass m and diameter $2b$, the other end of the rod being fixed.

(e) Prove that if a rigid body be moving under the action of external forces the sum of whose moments about a given line is zero throughout the motion, the angular momentum of the body about that line remains unchanged throughout the motion.

(f) When a body moves under the action of a system of conservative forces, the sum of its kinetic and potential energies is constant throughout the motion. Prove the statement.

4. Answer the following: 10×4=40

(a) (i) Show that the moment of inertia of semi-circular lamina about a tangent parallel to the bounding diameter is $Ma^2\left(\frac{5}{4}-\frac{8}{3\pi}\right)$, where a is the radius and M is the mass of the lamina. 5

(ii) Given moments and products of inertia about axes through the centre of gravity, find the moments and products of inertia about parallel axis. 5

Or

(a) (i) A lamina in the form of an ellipse is rotating in its own plane about one of its foci with angular velocity ω. This focus is set free and the other focus at the same instant is fixed. Show that the ellipse now rotates about it with angular velocity $\left(\frac{2-5e^2}{2+3e^2}\right)\omega$. 5

(ii) The lengths AB and AD of the sides of a rectangle $ABCD$ are $2a$ and $2b$ respectively. Show that the inclination to AB of one of the principal axes at A is

$$\frac{1}{2}\tan^{-1}\frac{3ab}{2\left(a^2-b^2\right)}.$$ 5

(b) (i) State and prove the D'Alembert's principle. 5

(ii) Find the length of the simple equivalent pendulum for an elliptic lamina when the axis is a latus rectum. 5

Or

(b) (i) Use Lagrange's equations to find the equation of motion of the compound pendulum which oscillates in a vertical plane about a fixed horizontal axis. 5

(ii) An elliptic lamina is rotating about its centre on a smooth horizontal table. If ω_1, ω_2, ω_3 be its angular velocities when the extremity of its axis, its focus and the extremity of its minor axis respectively become fixed, prove that

$$\frac{7}{\omega_1} = \frac{6}{\omega_2} + \frac{5}{\omega_3}.$$ 5

(c) (i) If $S \equiv Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy =$ constant be the equation of the momental ellipsoid at the centre of gravity O of a body referred to any rectangular axes through O, then prove that momental ellipsoid at the point (p, q, r) is

$S + M\left[(qz - ry)^2 + (rx - pz)^2 + (py - qx)^2\right] =$ constant, where M is the mass of the body. 5

(ii) A rough uniform board of mass m and length $2a$ rests on a smooth horizontal plane and a man of mass M walks on it from one end to the other. Show that the distance through which the board moves in this time is

$$\frac{2Ma}{M+m}.$$ 5

Or

(c) (i) A uniform rod of mass m and length $2a$ can turn freely about one end which is fixed. It is started with angular velocity ω from the position in which it hangs vertically. Find its angular velocity at any instant. 5

(ii) A uniform elliptic board swings about a horizontal axis at right angle to the board and passing through one focus. If the centre of oscillation be at the other focus, prove that the eccentricity of the ellipse is $\sqrt{2/5}$. 5

(d) A uniform solid sphere rolls down an inclined plane whose inclination to the horizontal is α. Show that the least co-efficient of friction between it and the plane, so that it may roll and not slide is $\frac{2}{7}\tan\alpha$. 10

Or

Derive Lagrange's equation in generalized co-ordinates.

OPTION–F

(Group Theory–II)

Paper : MAT–HE–6066

1. Answer the following as directed : $1 \times 10 = 10$

   (a) An isomorphim from a group to itself is called

   (i) endomorphism

   (ii) monomorphism

   (iii) automorphism

   (iv) None of the above

   (Choose the correct option)

   (b) A group of order 4 is isomorphic to z_4 or $z_2 \oplus z_2$. Is it true ?

   (c) Define internal direct product of *two* subgroups of a group.

   (d) List the elements of $\cup(8) \oplus \cup(10)$.

   (e) Let G be a group. $|Inn(G)| = 1$ if and only if ..... *(Complete the statement)*

(f) Is the conjugacy relation on a group an equivalence relation ?

(g) If G is a non-abelian group of order p^3, where p is a prime then $O(Z(G))$ is

(i) either 1 or p

(ii) either p or p^2

(iii) either p^2 or p^3

(iv) either 1 or p^3

(Choose the correct option)

(h) Define normalizer of an element a of a group G.

(i) State Sylow's third theorem.

(j) Let $H = \{(1), (1\,2)\}$. Is H normal in S_3?

2. Answer the following questions : 2×5=10

(a) If $f : G \to \overline{G}$ is an isomorphism from a group G onto a group $\overline{G}$, then prove that f carries the identity of G to the identity of $\overline{G}$.

(b) A group G is abelian if and only if the number of conjugate classes in G is same as the order of G. Is it true ? Justify your answer.

(c) Express Aut $(U(25))$ in the form $Z_m \oplus Z_n$.

(d) Determine all normal subgroups of D_n of order 2.

(e) Why is any abelian group of order 15 cyclic? Give reason.

3. Answer ***any four*** questions : 5×4=20

(a) Let H be a normal subgroup of a group G and K be any subgroup of G. Prove that HK is a subgroup of G.

(b) Let $Z(G)$ be the centre of a group G. If $G/Z(G)$ is cyclic then prove that G is abelian.

(c) Suppose ϕ is an isomorphism from a group G onto a group $\overline{G}$. If K is a subgroup of G then prove that $\phi(K) = \{\phi(K) \mid k \in K\}$ is subgroup of $\overline{G}$.

(d) Let G and H be finite groups. Prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime.

(e) Prove that a group of order p^2 is abelian, where p a prime.

(f) If m divides the order of a finite abelian group G, then prove that G has a subgroup of order m.

4. Answer the following questions : 10×4=40

(a) Define inner automorphism of a group.

Show that the set $I(G)$ of all inner automorphism of a group G is a subgroup of *Aut (G)*, where *Aut (G)* is the set of all automorphisms of G.

If T_{g_1} and T_{g_2} are *any two* inner auto-morphisms of a group G then show that $T_{g_1} = T_{g_2}$ if and only if $g_1 Z(G) = g_2 Z(G)$ where $Z(G)$ is the centre of G. 1+5+4=10

Or

State and prove Sylow's 2nd theorem. 10

(b) Let G be a group of order p^n, p a prime and n is a positive integer. Show that $O(Z(G)) > 1$ where $Z(G)$ is the centre of G.

Also determine the $O(Z(G))$, where G is a non-abelian group of order p^3. 7+3=10

Or

Prove that every p-subgroup of a finite group G is contained in some Sylow p-subgroups of G. 10

(c) Let G be a finite abelian group. Prove that G is isomorphic to the direct product of its Sylow subgroups. 10

Or

If a group G is the internal direct product of a finite number of subgroups $H_1, H_2,..., H_n$ then prove that G is isomorphic to the external direct product of $H_1, H_2....., H_n$.

(d) Let G be a finite abelian group and $p \mid O(G)$, where p is a prime. Show that there exists an element x in G such that $O(x) = p$. 10

Or

Determine the number of elements of order 5 in $Z_{25} \oplus Z_5$.

Prove that the order of an element in a direct product of a finite number of finite groups is the least common multiple of the orders of the components of the element. In symbols,

$$|(g_1, g_2,..., g_n)| = l.c.m(|g_1|, |g_2|,...., |g_n|).$$

4+6=10

OPTION–G

(Mathematical Finance)

Paper : MAT–HE–6076

1. Answer the following as directed: 1×10=10

   (a) "Forward contracts can be used to hedge foreign currency risk."

   (True ***or*** *False).*

   (b) Consider a stock that pays no dividend and is worth Rs. 60. If you can borrow or lend money for 1 year at 5%, what is the 1-year forward price of the stock ?

   (c) Write the name of the types of options.

   (d) What is arbitrage ?

   (e) Write the full form of OTC.

   (f) A combination of a stop order and a limit order is known as _____.

   (Fill in the blank)

   (g) Write the relation among the value of a swap, the floating-rate bond and the value of the fixed-rate bond.

   (h) What is implied volatility ?

(i) What is meant by the 'Rho' of a portfolio of options ?

(j) One index option contract is on ____ times the index. *(Fill in the blank)*

2. Answer the following : 2×5=10

(a) Explain the difference between hedging and speculation.

(b) Justify that CAPM is a pricing model.

(c) Define the term 'options'.

(d) Explain what you mean by the credit risk and the market risk in a financial contract.

(e) What are the formulas for u and d in terms of volatility ?

3. Answer ***any four*** parts : 5×4=20

(a) A debt of Rs. 55,000 is to be amortized over 9 years at 9% interest p.a. compounded annually. What value of monthly payments will achieve this ?

(b) Mr. Bori buys European put option with a strike price of Rs. 100 per share to purchase 100 shares of ABC Company after 4 months. Option price is Rs. 5 per share. If the price of one share is Rs. 98 on expiration date. What will be Mr. Bori's gain/loss if the option is exercised ? Should he exercise the option in this case ?

(c) Rs. 1,570 is invested at 12% p.a. compound interest. After how many years will the investment be worth Rs. 23,000 ?

(d) A call option on a non-dividend paying stock has a market price of Rs. 2.50. The stock price is Rs. 15, the exercise price is Rs. 13, the time to maturity is 3 months, and the risk-free interest rate is 5% per annum. What is the implied volatility ?

(e) What is meant by the gamma of an option position ? What are the risks in the situation where the gamma of a position is highly negative and the delta is zero ?

(f) Explain what is meant by

(i) the 3-month LIBOR rate;

(ii) the 3-month OIS rate.

Which is higher? Why?

4. Answer the following questions: 10×4=40

(a) Explain the following terms briefly: 2½×4=10

(i) Zero rates

(ii) Currency scoops

(iii) Put-call party

(iv) Types of traders

Or

Companies *A* and *B* have been offered the following rates per annum on a Rs. 2 lakh 5-year loan:

	Fixed rate	Floating rate
Company *A*	5.0%	LIBOR +0.1%
Company *B*	6.4%	LIBOR +0.6%

Company *A* requires a floating rate loan, Company *B* requires a fixed rate loan. Design a swap that will net a bank, acting as intermediary, 0.1% per annum and that will appear equally attractive to both companies. 10

(b) The spot price of silver is Rs. 25 per gram. The storage costs are Rs. 0.24 per gram per year payable quarterly in advance. Assuring that interest rates are 5% per annum for all maturities, calculate the futures price of silver for delivery in 9 months. 10

Or

An investor receives Rs. 1100 in one year in return for an investment of Rs. 1000 now. Calculate the percentage return per annum with

(i) annual compounding;

(ii) semi-annual compounding;

(iii) monthly compounding;

(iv) continuous compounding.

(c) What are the most important aspects of the design of a new futures contract? Explain how margin accounts protect investors against the possibility of default. 10

Or

A stock index is currently 300, the dividend yield on the index is 3% per annum, and the risk-force interest is 8% per annum. What is a lower bound for the price of a six-month European call option on the index when the strike price is 290 ?

(d) What does the Black-Scholes-Merton stock option pricing model assume about the probability distribution of the stock price in one year ? What does it assume about the probability distribution of the continuously compounded rate of return on the stock during the year ? 10

Or

Let $C(k, t)$ be the cost of a cell option on a specified security that has strike price k and expiration time t.

Prove that

(i) for fixed expiration time t, $C(k,t)$ is a convex and nonincreasing function of k; 5

(ii) for $s > 0$, $C(k, t) - C(k + s, t) \leq se^{-rt}$. 5