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3 (Sem-2/CBCS) MAT HC 1

2023

**MATHEMATICS**

(Honours Core)

Paper : MAT-HC-2016

**(Real Analysis)**

Full Marks : 80

Time : Three hours

**The figures in the margin indicate full marks for the questions.**

1. Answer the following questions as directed :  
1×10=10

(a) Give an example of a set which is not bounded below.

(b) Write the completeness property of  $\mathbb{R}$ .

(c) If  $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ , then what will be  $\inf S$  ?

Contd.

(d) The unit interval  $[0, 1]$  in  $\mathbb{R}$  is not countable.

(State whether True or False)

(e) Define a convergent sequence of real numbers.

(f) What is the limit of the sequence.  $\{x_n\}$ ,

$$\text{where } x_n = \frac{5n+2}{n+1}, n \in \mathbb{N} ?$$

(g) A bounded monotone sequence of real numbers is convergent.

(State whether True or False)

(h) What is the value of  $r$  if the geometric

$$\text{series } \sum_{n=0}^{\infty} r^n \text{ is convergent ?}$$

(i) The series  $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots$  is not convergent.

(State whether True or False)

(j) If  $\sum_{n=1}^{\infty} u_n$  is a positive term series such

that  $\lim_{n \rightarrow \infty} (u_n)^{1/n} = l$ , then the series converges, if

(i)  $l < 1$

(ii)  $0 < l < 2$

(iii)  $l > 1$

(iv)  $1 \leq l < 2$

(Choose the correct option)

2. Answer the following questions :  $2 \times 5 = 10$

(a) Find the supremum of the set

$$S = \{x \in \mathbb{R} : x^2 - 3x + 2 < 0\}.$$

(b) If  $(x_n)$  and  $(y_n)$  are convergent sequences of real numbers and  $x_n \leq y_n \forall n \in \mathbb{N}$ , then show that

$$\lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} y_n.$$

(c) Show that the sequence  $((-1)^n)$  is divergent.

(d) Define absolutely convergent series and give an example.

(e) Show that the series  $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$  is convergent.

3. Answer **any four** questions :  $5 \times 4 = 20$

(a) Prove that if  $x \in \mathbb{R}$ , then there exists  $n_x \in \mathbb{N}$  such that  $x \leq n_x$ .

(b) If  $x$  and  $y$  are real numbers with  $x < y$ , then show that there exists an irrational number  $z$  such that  $x < z < y$ .

(c) Show that if a sequence  $(x_n)$  of real numbers converges to a real number  $x$ , then any subsequence of  $(x_n)$  also converges to  $x$ .

(d) Show that the sequence

$\left( (-1)^n + \frac{1}{n} \right)$ ,  $n \in \mathbb{N}$  is not a Cauchy sequence.

- (e) Using ratio test establish the convergence or divergence of the series whose  $n$ th term is  $\frac{n!}{n^n}$ .
- (f) Let  $z = (z_n)$  be a decreasing sequence of strictly positive numbers with  $\lim(z_n) = 0$ . Prove that the alternating series  $\sum (-1)^{n+1} z_n$  is convergent.

4. Answer the following questions :  $10 \times 4 = 40$

- (a) Prove that the set  $\mathbb{R}$  of real numbers is not countable.

**Or**

If  $S$  is a subset of  $\mathbb{R}$  that contains at least two points and has the property : if  $x, y \in S$  and  $x < y$ , then  $[x, y] \subseteq S$ , then show that  $S$  is an interval.

- (b) Prove that a sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Or

Let  $(x_n)$  be a sequence of positive real numbers such that  $L = \lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n}$  exists.

If  $L < 1$ , then show that  $(x_n)$  converges and  $\lim_{n \rightarrow \infty} x_n = 0$ .

(c) (i) Show that  $\lim \left( \frac{1}{n^2 + 1} \right) = 0$  2½

(ii) Show that the sequence  $\left( \frac{1}{n} \right)$  is a Cauchy sequence. 2½

(iii) Prove that every contractive sequence is a Cauchy sequence. 5

Or

State and prove the monotone subsequence theorem. 10

(d) Prove that a positive term series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is convergent if  $p > 1$  and divergent if  $0 < p \leq 1$ .

Or

Show that a necessary condition for convergence of an infinite series  $\sum_{n=1}^{\infty} u_n$

is that  $\lim_{n \rightarrow \infty} u_n = 0$ . Demonstrate by an example that this is not a sufficient condition for the convergence.