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3 (Sem-3/CBCS) MAT HC 2

2022

MATHEMATICS

(Honours)

Paper : MAT-HC-3026

(Group Theory-I)

Full Marks : 80

Time : Three hours.

The figures in the margin indicate full marks for the questions.

1. Answer **any ten** questions : $1 \times 10 = 10$

(a) What do you mean by the symmetry group of a plane figure ?

(b) The set S of positive irrational numbers together with 1 is a group under multiplication. Justify whether it is true **or** false.

Contd.

- (c) Define a binary operation on the set $\{0, 1, 2, 3, 4, 5\}$ for which it is a group.
- (d) Let $G = \langle a \rangle$ be a cyclic group of order n . Write a necessary and sufficient condition for which a^k is a generator of G .
- (e) What do you mean by even permutation? Give an example.
- (f) Write the order of the alternating group of degree n .
- (g) Let $G = S_3$ and $H = \{(1), (13)\}$. Write the left cosets of H in G .
- (h) Show that there is no isomorphism from Q , the group of rational numbers under addition, to $Q^\#$, the group of non-zero rational numbers under multiplication.
- (i) State Cayley's theorem.
- (j) Let $\phi : \mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$ be defined by $\phi(x) = 3x$, $x \in \mathbb{Z}_{12}$. Find $\ker \phi$.

- (k) On the set $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$, define a binary operation for which it is a group.
- (l) Define normalizer of an element in a group G .
- (m) Product of two subgroups of a group is again a subgroup. State whether true **or** false.
- (n) State Lagrange's theorem.
- (o) What is meant by external direct product of a finite number of groups?
- (p) Find the order of the permutation

$$f = \begin{pmatrix} a & b & c & d & e \\ c & a & b & e & d \end{pmatrix}$$

- (q) The subgroup of an abelian group is abelian. State whether it is true **or** false.
- (r) Give the statement of third isomorphism theorem.

2. Answer **any five** questions : $2 \times 5 = 10$

(a) Show that in a group G , right and left cancellation laws hold.

(b) Show that a group of prime order is cyclic.

(c) Every subgroup of an abelian group is normal. Justify whether it is true **or** false.

(d) Let \mathbb{C}^* denote the group of non-zero complex numbers under multiplication. Define $\phi: \mathbb{C}^* \rightarrow \mathbb{C}^*$ by $\phi(x) = x^4, x \in \mathbb{C}^*$. Show that ϕ is a homomorphism and find $\ker \phi$.

(e) If ϕ is an isomorphism from a group G onto a group \bar{G} , then show that ϕ carries the identity element of G to the identity element of \bar{G} .

(f) What is meant by cycle of a permutation? Give an example.

(g) Show that in a group (G, \bullet) ,

$$(a.b)^{-1} = b^{-1}.a^{-1}, \quad a, b \in G.$$

(h) Define centre of a group G and give an example.

(i) Give an example of a group containing only three elements.

(j) Define group isomorphism and give an example.

3. Answer **any four** questions : 5×4=20

(a) Show that *any two* cycles of a permutation of a finite set are disjoint.

(b) If H and K are two normal subgroups of a group G such that $H \cap K = \{e\}$ (e being the identity element of G), then show that $hk = kh$ for all $h \in H, k \in K$.

(c) Let H be a subgroup of a group G . Show that there exists a one-one and onto map between the set of all left cosets of H in G and the set of all right cosets of H in G .

- (d) Let G be a group. If $a \in G$ is of finite order n and also $a^m = e$, then show that n/m .
- (e) Let f be a homomorphism from a group G to a group G' . Show that $\ker f$ is a normal subgroup of G .
- (f) If \mathbb{R}^* is the group of non-zero real numbers under multiplication, then show that (\mathbb{R}^*, \cdot) is not isomorphic to $(\mathbb{R}, +)$.
- (g) Prove that a cyclic group is abelian.
- (h) Consider the multiplicative group $G = \{1, -1, i, -i\}$. Define a self mapping ϕ on G which is a homomorphism and justify your answer.

4. Answer **any four** questions : $10 \times 4 = 40$

(a) Let G be a group. Show that

(i) the centre of G is a subgroup of G ;

(ii) for each $a \in G$, the centralizer of a is a subgroup of G .

(b) Let G be a group in which

$$(ab)^3 = a^3b^3$$

$$(ab)^5 = a^5b^5 \text{ for all } a, b \in G.$$

Prove that G is abelian.

(c) Prove that every subgroup of a cyclic group is cyclic. Also show that if

$\langle a \rangle = n$, then the order of any subgroup

of $\langle a \rangle$ is a divisor of n .

(d) If H and K are finite subgroups of a group G , then prove that

$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|}$$

- (e) Prove that the order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles.
- (f) Let G be a finite abelian group and let p be a prime that divides the order of G . Prove that G has an element of order p .
- (g) Let ϕ be an isomorphism from a group G onto a group \bar{G} . Prove that —
- (i) for every integer n and for every $a \in G$, $\phi(a^n) = [\phi(a)]^n$;
- (ii) $|a| = |\phi(a)|$ for all $a \in G$.
- (h) State and prove the second isomorphism theorem for groups.
- (i) Show that the order of a cyclic group is same as the order of its generator.
- (j) Consider the multiplicative group $G = \{1, -1, i, -i\}$. Find all the subgroups of G and verify Lagrange's theorem for each subgroup.