2019

MATHEMATICS

(Major)

Paper : 3.1

(Abstract Algebra)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following as directed : 1×10=10

(a) Define kernel of a group homomorphism.

(b) A one-one homomorphism from a group G onto itself is called

(i) epimorphism

(ii) monomorphism

(iii) endomorphism

(iv) None of the above

(Choose the correct option)

(c) If $f : G \to G'$ is a group homomorphism, then the set of inverse images under f of elements of a normal subgroup of G' is normal in G.

(State True or False)

(d) The ring of all 2×2 matrices over reals under matrix addition and multiplication is an integral domain.

(State True or False)

(e) A ring R is commutative if and only if

(i) $Z(R)$ is an ideal of R

(ii) $Z(R)$ is a subring of R

(iii) $Z(R) = R$

(iv) $Z(R) \subset R$

(Choose the correct option)

(f) Define characteristic of a ring R.

(g) A group G is Abelian if and only if the number of conjugate classes in G is same as order of G.

(State True or False)

(h) Let G be a finite group of order 36. If H is a Sylow 3-subgroup of G, then which of the following is possible?

(i) $o(H) = 3$

(ii) $o(H) = 9$

(iii) $o(H) = 18$

(iv) All of the above

(Choose the correct option)

(i) Define normalizer of an element of a group.

(j) Quotient ring of an integral domain is again an integral domain.

(State True or False)

2. Answer the following questions : $2\times5=10$

(a) If $f : G \to G'$ is a group homomorphism, then show that $f(e) = e'$, where e and e' are identities of G and G' respectively.

(b) Give example of subring of a ring which is not an ideal of the ring.

(c) Show that if G is a non-Abelian group, then the map $f : G \to G$ defined by $f(x) = x^{-1}$, $\forall\ x \in G$ is not an automorphism.

(d) Give example of a quotient ring $\frac{R}{S}$, such that R is not an integral domain but $\frac{R}{S}$ is an integral domain.

(e) Define Euclidean domain.

3. Answer any *four* questions : 5×4=20

(a) Show that the group of all non-zero complex numbers under multiplication of complex numbers is isomorphic to the group of all 2×2 real matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where both a and b are not zero, under matrix multiplication.

(b) If A, B, C are ideals of a ring R, such that $B \subseteq A$, then prove that

$$A \cap (B + C) = B + (A \cap C)$$

(c) Determine the automorphism group Aut(G), where G is an infinite cyclic group.

(d) Prove that every ideal in a Euclidean domain is a principal ideal.

(e) If A and B are two ideals of a ring R, then prove that

$$\frac{A+B}{B} \cong \frac{A}{A \cap B}$$

(f) Define conjugate relation on a group G and show that it is an equivalence relation on G.

4. Answer the following questions : 10×4=40

(a) Let $f: G \to G'$ be an epimorphism from the group G onto the group G' and H' be a subgroup of G', then prove that—

(i) $H = \{x \in G : f(x) \in H'\}$ is a subgroup of G containing ker f;

(ii) H is normal subgroup of G if and only if H' is normal in G';

(iii) if H' is normal in G', then

$$\frac{G'}{H'} \cong \frac{G}{H}$$

2+3+5=10

Or

If H is any subgroup of the group G and N is a normal subgroup of G, then prove that—

(i) $H \cap N$ is a normal subgroup of G;

(ii) N is a normal subgroup of

$$HN = \{hn : h \in H, n \in N\};$$

(iii) $\dfrac{HN}{N} \cong \dfrac{H}{H \cap N}$. 2+2+6=10 *(b)* Define maximal ideal of a ring R. Show that $H_4 = \{4n : n \in Z\}$ is a maximal ideal of the ring of even integers $(E, +, .)$. Is H_4 a prime ideal of $(E, +, .)$? Justify. Prove that in a Boolean ring R, every prime ideal $P (\neq R)$ is maximal.

1+4+1+4=10

Or

If W be a subspace of the vector space $V(F)$, then show that the set

$$\frac{V}{W} = \{W + v : v \in V\}$$

forms a vector space over F. 10

(c) Let G be a finite Abelian group and $p \mid o(G)$, where p is a prime, then show that there exists an element x in G such that $o(x) = p$. 10

Or

For any finite group G, show that the set Aut(G) of all automorphisms on G is a subgroup of the group $A(G)$ of all permutations on the set G. Also show that the set $I(G)$ of all inner automorphisms on G is a subgroup of Aut(G).

6+4=10

(d) If I is an ideal of the ring R, then show that the set $\frac{R}{I}$ of all cosets of I in R forms a ring. Also show that the relation ~ defined on R by $a \sim b \Leftrightarrow a - b \in I$, $\forall\ a, b \in R$ is an equivalence relation and each $a + I \in \frac{R}{I}$ represents an equivalence class arising from the equivalence relation ~. 5+3+2=10

Or

Show that any ring can be embedded into a ring with unity. 10

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