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3 (Sem–4/CBCS) MAT HC 1

2024

MATHEMATICS

(Honours Core)

Paper : MAT–HC–4016

(Multivariate Calculus)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1\times10=10$

   (a) If $f(x, y) = ln(y - x)$, then find domain of it.

   (b) Define level curve of a function $f(x, y)$ at a constant C.

   (c) Find f_x if $f(x, y) = (\sin x^2)\cos y$.

Contd.

(d) If $f(x, y) = \sin xy$, then df is

(i) $y\cos xy\,dx + x\cos xy\,dy$

(ii) $y\cos xy\,dy + x\cos xy\,dx$

(iii) $y\cos x\,dx + x\cos y\,dy$

(iv) $\cos xy\,dx + \cos xy\,dy$

(Choose the correct option)

(e) Evaluate $\frac{\partial(x, y)}{\partial(r, \theta)}$ for the transformation $x = r\cos\theta,\ y = r\sin\theta$.

(f) If $P_0(x_0, y_0)$ is a critical point of $f(x, y)$ and f has continuous 2nd order partial derivatives in a disk centered at (x_0, y_0) and $D = f_{xx}f_{yy} - f_{xy}^2$, then a relative minimum occurs at P_0, if

(i) $D(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) < 0$

(ii) $D(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) > 0$

(iii) $D(x_0, y_0) < 0$ and $f_{yy}(x_0, y_0) < 0$

(iv) $D(x_0, y_0) < 0$ and $f_{yy}(x_0, y_0) > 0$

(Choose the correct option)

(g) The curl of a vector field

$$V(x, y, z) = u(x, y, z)i + v(x, y, z)j + w(x, y, z)k$$ is

(i) $\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)i + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)j + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)k$

(ii) $\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)i + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)k$

(iii) $\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right)i + \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)j + \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)k$

(iv) $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)i + \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right)j + \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)k$

(Choose the correct option)

(h) Define work as a line integral.

(i) State Green's theorem on a simply connected region D.

(j) If a vector field F and curl F are both continuous in a simply connected region D on $\mathbb{R}^3$, then F is conservative in D if curl $F \neq 0$. State whether this statement is true *or* false.

2. Answer the following questions : $2\times5=10$

(a) Show that the function f is continuous at (0, 0) where

$$f(x, y) = \begin{cases} y \sin\frac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

(b) Compute the slope of the tangent line to the graph of $f(x, y) = x^2 \sin(x + y)$ at the point $P\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$ in the direction parallel to the XZ plane.

(c) Evaluate $\iint\limits_R \frac{2xy}{x^2+1}\, dA$ where $0 \le x \le 1$, $1 \le y \le 3$.

(d) Show that $div\ F = 0$ and $curl\ F = 0$, if F is a constant vector field.

(e) Evaluate $\int_0^2 \int_0^1 \int_{-1}^2 8x^2yz^3\, dx\, dy\, dz$.

3. Answer ***any four*** questions : 5×4=20

(a) (i) Find $\frac{\partial w}{\partial r}$ where $w = e^{2x-y+3z^2}$ and $x = r + s - t,\ y = 2r - 3s,\ z = \cos rst$. 3

(ii) Show that $\lim\limits_{(x,\,y)\to(0,\,0)} \frac{x+y}{x-y}$ does not exist. 2

(b) (i) If f has a relative extremum at $P_0(x_0, y_0)$ and both f_x and f_y exist at (x_0, y_0), then prove that $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. 2

(ii) Discuss the nature of the critical points of the function $f(x, y) = (x-2)^2 + (y-3)^4$. 3

(c) Use a polar double integral to show that a sphere of radius a has volume $\frac{4}{3}\pi a^3$.

(d) An object moves in the force field $F = y^2 i + 2(x+1)yj$. How much work is performed as the object moves from the point (2, 0) counterclockwise along the elliptical path $x^2 + 4y^2 = 4$ to (0, 1), then back to (2, 0) along the line segment joining the two points.

(e) (i) Evaluate $\int_C (x^2 + y^2)dx + 2xy\,dy$ where C is the quarter circle $x^2 + y^2 = 1$ from (1, 0) to (0, 1). 3

(ii) A wire has the shape of the curve $x = \sqrt{2}\sin t,\ y = \cos t,\ z = \cos t$ for $0 \le t \le \pi$. If the wire has density $\delta(x, y, z) = xyz$ at each point (x, y, z), find its mass. 2

(f) Find the volume of the solid in the first octant that is bounded by the cylinder $x^2 + y^2 = 2y$, the half cone $z = \sqrt{x^2 + y^2}$ and the xy-plane.

4. Answer the following questions : 10×4=40

(a) (i) Let $f(x, y)$ be a function that is differentiable at $P_0(x_0, y_0)$. Prove that f has a directional derivative in the direction of the unit vector $u = u_1 i + u_2 j$ given by

$$D_u f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

3

(ii) Find the directional derivative of $f(x, y) = ln(x^2 + y^2)$ at $P_0(1, -3)$ in the direction of $v = 2i - 3j$ using the gradient formula. 3

(iii) Find the equations of the tangent plane and the normal line to the cone $z^2 = x^2 + y^2$ at the point where $x = 3, y = 4$ and $z > 0$.

4

OR

(i) Prove that if $f(x, y)$ is differentiable at (x_0, y_0), then it is continuous there. 4

(ii) When two resistances R_1 and R_2 are connected in parallel, the total resistance R satisfies $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$.

If R_1 is measured as 300 ohms with a maximum error of 2% and R_2 is measured as 500 ohms with a maximum error of 3%, then find the maximum percentage error in R. 6

(b) (i) Use the method of Lagrange multipliers to minimize $f(x, y)=x^2-xy+2y^2$ subject to $2x+y=22$. 5

(ii) Find all relative extrema and saddle points on the graph of $f(x, y)=x^2y^4$. 5

OR

(i) Find the absolute extrema of the function $f(x, y)=e^{x^2-y^2}$ over the disk $x^2+y^2\leq 1$. 6

(ii) Suppose E be an extreme value of f subject to the constraint $g(x, y) = C$. Prove that the Lagrange multiplier λ is the rate of change of E with respect to C. 4

(c) (i) Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\, dy\, dx$ by converting to polar coordinates. 5

(ii) Evaluate $\iiint_D e^z\, dv$ where D is the region described by the inequalities $0 \le x \le 1$, $0 \le y \le x$ and $0 \le x \le x + y$. 5

OR

(i) Find the volume of the solid bounded above by the plane $z = y$ and below in the xy-plane by the part of the disk $x^2 + y^2 \le 1$ in the 1st quadrant. 5

(ii) Evaluate : $\iiint_D x\,dV$ where D is the solid in the 1st octant bounded by the cylinder $x^2+y^2=4$ and the plane $2y+z=4$. 5

(d) (i) Let C be a piecewise smooth curve that is parameterized by a vector function $R(t)$ for $a \le t \le b$ and let F be a vector field that is continuous on C. If f is a scalar function such that $F = \nabla f$, then prove that $\int_C F.dR = f(Q) - f(P)$ where $Q = R(b)$ and $P = R(a)$ are the end points of C.

Using it evaluate the line integral $\int_C F.dR$, where

$F = \nabla\left(e^x \sin y - xy - 2y\right)$ and C is the path described by

$$R(t) = \left[t^3 \sin\frac{\pi}{2}t\right] i - \left[\frac{\pi}{2}\cos\left(\frac{\pi}{2}t + \frac{\pi}{2}\right)\right] j$$

for $0 \le t \le 1$ 5+3=8

(ii) Determine whether the vector field $F(x, y) = \frac{(y+1)i - xj}{(y+1)^2}$ is conservative. 2

OR

(i) Evaluate $\oint_C \left(\frac{1}{2}y^2dx + zdy + xdz\right)$ where C is the curve of intersection of the plane $x + z = 1$ and the ellipsoid $x^2 + 2y^2 + z^2 = 1$, oriented counterclockwise as viewed from above. 6

(ii) Evaluate $\iint_S F.NdS$ where $F = x^2i + xyj + x^3y^3k$ and S is the surface of the tetrahedron bounded by the plane $x + y + z = 1$ and the coordinate planes with outward unit normal vector N. 4