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3 (Sem–5/CBCS) MAT HC 2

2024

MATHEMATICS

(Honours Core)

Paper : MAT–HC–5026

(Linear Algebra)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : 1×10=10

(a) Give reason why a line in $\mathbb{R}^2$ not passing through the origin is not a subspace of $\mathbb{R}^2$.

(b) Express $W = \left\{ \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix}; a,b \in \mathbb{R} \right\}$

as span of two vectors.

Contd.

(c) State whether the fallowing statement is true *or* false :

"A finite dimensional vector space has exactly one basis."

(d) Find the dimension of the subspace of all vectors in $\mathbb{R}^3$ whose first and third entries are equal.

(e) 0 is an eigenvalue of a matrix A if and only if A is ______. *(Fill in the blank)*

(f) When is a square matrix said to be diagonalizable?

(g) Write the kernel of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(x,y,z) = (x,0,z)$.

(h) If $u = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$, then compute $u.v$.

(i) What is the distance between the vectors $\vec{u} = (7,1)$ and $\vec{v} = (3,2)$ in the $\mathbb{R}^2$ plane ?

(j) What do you mean by orthogonal vectors in an inner product space?

2. Answer the following questions: $2\times5=10$

(a) Let $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Determine if w is in null space of A.

(b) Let $\mathbb{P}_3$ be the vector space of all polynomials of degree atmost 3.

Are the vectors

$p(t) = 1 + t^2$ and $q(t) = 1 - t^2$ linearly independent in $\mathbb{P}_3$? Justify your answer.

(c) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 4 & -3 \end{bmatrix}$$

(d) Let $\mathcal{B} = \{b_1, b_2, b_3\}$ be a basis for a vector space V and $T : V \rightarrow \mathbb{R}^2$ be a linear transformation such that

$$T(x_1b_1 + x_2b_2 + x_3b_3) = \begin{bmatrix} 2x_1 - 4x_2 + 5x_3 \\ -x_2 + 3x_3 \end{bmatrix}.$$

Find the matrix for T relative to $\mathcal{B}$.

(e) Let $v=(1,-2,2,0)$ be a vector in $\mathbb{R}^4$. Find a unit vector u in the same direction as v.

3. Answer ***any four*** questions : 5×4=20

(a) If a vector space V has a basis $\mathcal{B}=\{b_1,b_2,\ldots,b_n\}$, then prove that any set in V containing more than n vectors must be linearly dependent.

(b) Let $b_1=\begin{bmatrix}1\\-3\end{bmatrix}, b_2=\begin{bmatrix}2\\-5\end{bmatrix}$ and $x=\begin{bmatrix}-2\\1\end{bmatrix}$.

3+2=5

(i) Show that the set $\mathcal{B}=\{b_1,b_2\}$ is a basis of $\mathbb{R}^2$

(ii) Find the coordinate vector $[x]$ of x relative to $\mathcal{B}$.

(c) Given that 2 is an eigenvalue of the matrix $A=\begin{bmatrix}4 & -1 & 6\\2 & 1 & 6\\2 & -1 & 8\end{bmatrix}$

Find a basis for the corresponding eigenspace.

(d) Prove that an $n\times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

(e) Compute the orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin.

(f) If $\{u, v\}$ is an orthonormal set in an inner product space V, then show that $\|u - v\| = \sqrt{2}$.

Answer either *(a)* **or** *(b)* from each of the following questions : 10×4=40

4. (a) Find the rank and the nullity of the matrix 10

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

(b) Let $b_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $c_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ and consider the bases for $\mathbb{R}^2$ given by $\mathcal{B} = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ 5+5=10

(i) Find the change-of-coordinates matrix from C to $\mathcal{B}$

(ii) Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$

5. (a) (i) If $n \times n$ matrices A and B are similar, then show that they have the same characteristic polynomial. 3

(ii) If $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ then find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$ 7

(b) (i) Let $\mathcal{B} = \{b_1, b_2, b_3\}$ and $\mathcal{D} = \{d_1, d_2\}$ be bases for vector spaces V and W respectively. Let $T : V \to W$ be a linear transformation with the property that

$$T(b_1) = 3d_1 - 5d_2,\ T(b_2) = -d_1 + 6d_2,\ T(b_3) = 4d_2.$$

Find the matrix for T relative to $\mathcal{B}$ and $\mathcal{D}$. 5

(ii) Let A be a real 2×2 matrix with a complex eigenvalue

$\lambda = a - bi (b \neq 0)$ and an associated eigenvector v in $\mathbb{C}^2$. Show that

$A(Re\,v) = a\,Re\,v + b\,Im\,v$ and

$A(Im\,v) = b\,Re\,v + a\,Im\,v$ 5

6. (a) Let, $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right\}$ be a basis for a subspace W of $\mathbb{R}^3$. Using the Gram-Schmidt process construct an orthogonal basis for W.

Hence, find an orthonormal basis.

8+2=10

(b) What do you mean by an inner product on a vector space V? Consider the inner product in $\mathbb{R}^2$ defined by $\langle u,v\rangle = 4u_1v_1 + 5u_2v_2$ where $u=(u_1,u_2)$, $v=(v_1,v_2)\in\mathbb{R}^2$. If $x=(1,1)$ and $y=(5,-1)$, then find $\|x\|, \|y\|$ and $|\langle x,y\rangle|^2$. Also, show that in an inner product space V over $\mathbb{R}$,

$$\langle u,v\rangle = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2, \forall u,v\in V.$$

2+3+5=10

7. (a) State Cayley-Hamilton theorem for matrices. Verify the theorem for the matrix. 2+6+2=10

$$M = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and hence find M^{-1}.

(b) If possible, diagonalize the symmetric matrix 10

$$A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$