

1 (Sem-5/FYUGP) MAT 45 MN/(B)

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MATHEMATICS

(Minor)

Paper : MAT0500504

(Multivariate Calculus)

(Set-B)

Full Marks : 60

Time : 2½ hours

*The figures in the margin indicate full marks
for the questions.*

1. Answer any *eight* of the following : 1×8=8

(a) Find the domain of $f(x, y) = \frac{xy}{x^4 + y^4}$.

(b) If $f(x, y) = x \sin(xy)$, then find f_y .

(c) Define gradient of $f(x, y)$.

- (d) When is the function $f(x, y)$ said to have relative maximum at the point (x_0, y_0) of its domain?
- (e) What is the curl of a constant vector field?
- (f) Define the directional derivative of the function $f(x, y)$ at $P(x_0, y_0)$ in the direction of $\vec{u} = u_1\hat{i} + u_2\hat{j}$.
- (g) Define a vector field.
- (h) Find the curl $\vec{F}$, where $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$.
- (i) Define a critical point of a function f defined on an open set D .
- (j) Determine the constant a such that $\vec{\nabla} \cdot \vec{f} = 0$, where $\vec{f} = x\hat{i} + y\hat{j} + (x + az)\hat{k}$.

2. Answer any six of the following : 2×6=12

- (a) If $f(x, y) = e^y \cos x$, then find

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

- (b) Find the gradient of the function

$$f(x, y) = x^2y + y^3$$

(c) Find the critical point of

$$f(x, y) = (x-2)^2 + (y-3)^2$$

(d) Prove that $\lim_{\substack{x \rightarrow 1 \\ x \rightarrow 2}} (x^2 + 2y) = 5$.

(e) If $\vec{A} = xz\hat{i} - y^2\hat{j} + 2x^2y\hat{k}$, then find $\text{curl } \vec{A}$.

(f) Find the Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$, when $x = u + 2v, y = 3u - 4v$.

(g) Find the domain and range of the function $f(x, y) = \sqrt{9 - x^2 - y^2}$.

(h) If $x^2z + yz^3 = x$, then find $\frac{\partial z}{\partial x}$.

(i) Define absolute maximum and absolute minimum of a function $f(x, y)$ at the point (x_0, y_0) in the domain D .

(j) Find the directional derivative of $f(x, y) = \log(x^2 + 3y)$ at $(1, 1)$ in the direction of $\hat{i} + \hat{j}$.

3. Answer any *four* of the following :

5×4=20

(a) Show that

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 y}{x^4 + y^2}$$

does not exist.

(b) Use the method of Lagrange's multiplier to

$$\begin{aligned} &\text{maximize } f(x, y) = 16 - x^2 - y^2 \\ &\text{subject to } x + 2y = 6. \end{aligned}$$

(c) By direct calculation, show that

$$f_{xyz} = f_{yzx} = f_{zyx}$$

for the function

$$f(x, y, z) = xyz + x^2 y^3 z^4$$

(d) If $z = \sqrt{x^2 + 2xy}$, where $x = \cos \theta$,
 $y = \sin \theta$, then find $\frac{dz}{d\theta}$.

(e) In what direction is the function defined by $f(x, y) = xe^{2y-x}$ increasing most rapidly at the point $P_0(2, 1)$ and what is the maximum rate of increase?

(f) Find the equations for the tangent plane and normal line to the cone $z^2 = x^2 + y^2$ at the point where $x = 3$, $y = 4$ and $z > 0$.

(g) Find all relative extrema and saddle points of the function

$$f(x, y) = x^2 - 6x + 2y^2 + 4y - 2$$

(h) Show that $\text{grad } \phi$ is a vector perpendicular to the surface $\phi(x, y, z) = c$, where c is a constant.

4. Answer any *two* of the following questions :

10×2=20

(a) (i) If $u = f(x+2y) + g(x-2y)$, then show that

$$4 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

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(ii) Find the critical points of the function

$$f(x, y) = 2x^2 + 2xy + y^2 - 2x - 2y + 5$$

Also find all relative extrema.

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- (b) Define second-order partial derivative test for the extremum of the function $f(x, y)$ at (x_0, y_0) . By using the second-order partial test to classify each point as a relative extremum or a saddle point for the function $f(x, y) = 8x^3 - 24xy + y^3$. 10

- (c) (i) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2\hat{i} - \hat{j} - 2\hat{k}$. 5

- (ii) Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$. 5

- (d) (i) Investigate the continuity of $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$, $(x, y) \neq (0, 0)$
 0 , $(x, y) = (0, 0)$
 at $(0, 0)$. 5

- (ii) Find $\frac{\partial w}{\partial s}$, if $w = 4x + y^2 + z^3$, where $x = e^{rs^2}$, $y = \log \frac{r+s}{t}$, $z = rst^2$. 5

(7)

- (e) Find the point on the intersection of the plane $x + 2y + z = 10$ and the paraboloid $z = x^2 + y^2$ that is closest to the origin. 10

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