2 0 1 2

MATHEMATICS

(Major)

Paper : 3.1

(Abstract Algebra)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : 1×10=10

(a) Let f be a mapping from the additive group of integers to the group $G = \{-1, 1\}$ under multiplication, defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is even} \\ -1, & \text{if } x \text{ is odd} \end{cases}$$

Then which of the following statements is true?

(i) f is not a homomorphism

(ii) f is an onto homomorphism, but not one-one

(iii) f is an isomorphism

(iv) f is a homomorphism, but not one-one and onto

(b) What is the natural homomorphism from a group onto its quotient group?

(c) State Cayley's theorem for groups.

(d) Give the example of an infinite integral domain which is not a field.

(e) What is a simple ring?

(f) Let $R = \{0, 1\}$ mod 2. What is the characteristic of R?

(g) State whether the following statement is true or false :

> Let G be a non-Abelian group. Then the map $\theta : G \rightarrow G$ given by $\theta(x) = x^{-1}$ is an automorphism of G.

(h) Let G be a group and $a \in G$. What is the normalizer of a in G?

(i) Define Euclidean ring.

(j) State whether the following statement is true or false :

> A group of order p^2, where p is a prime, is Abelian.

2. Answer the following questions : $2\times5=10$

(a) Let G and G' be two groups and $f: G \rightarrow G'$ be a homomorphism. Show that f is one-one if $\ker f = \{e\}$, where e is the identity element in G.

(b) The union of two
space is again
whether it is true

(c) What do you
permutations? G
similar permutatio

(d) Let $R[x]$ be the rin
ring R. Show that
then $R[x]$ is also c

(e) Show by an examp
R_1 and R_2, we can
$f: R_1 \rightarrow R_2$ such th
of R_2, whereas 1

3. Answer the following q

(a) Let $f: G \rightarrow G'$ be a
from group G to G'
of G and H' be
Show that $f^{-1}(H$
G containing $\ker f$

(b) Find all the idem
elements of the rin

Or

If in a ring R with
$(xy)^2 = x^2$
then show that R i

(c) Let G be a group and $a \in G$ has only two conjugates in G. Show that $N(a)$ is a normal subgroup of G.

Or

Prove that a group of order 30 has either a normal subgroup of order 5 or a normal subgroup of order 3.

(d) Show that a ring homomorphism maps the zero element onto the zero element. Let

$$R_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in R \right\}$$

where R is a ring. Prove that the mapping

$$f : R_1 \to R,\ f\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = a$$

is an isomorphism between the rings R_1 and R.

4. Answer the following questions : 10×4=40

(a) Prove that the set A_n of all even permutations of S_n $(n \geq 2)$ is a normal subgroup of S_n and $o(A_n) = \dfrac{o(S_n)}{2}$. Find all the normal subgroups of S_4. 7+3

(Continued)

Or

Prove the fundamental
homomorphism. Show
cyclic group is isomorph
group of integers.

(b) Prove that an ideal M
ring R with unity is
$\Leftrightarrow R/M$ is a field. E
$Z/<4>$ is a field or no

Or

Let R be a finite (n
domain. Prove that $o(R)$
a prime. Find all the idea
(with usual notation).

(c) *(i)* State Sylow's 1st
theorems.

(ii) Define inner auto
group G. Prove th
inner automorphis
group of Aut G, wh
group of all autom

Or

Prove that the nur
of the conjugacy cl
finite group G is $o($
$N(a)$ is the normal
denotes the centre
that

$o(G) = o(Z(G)) +$

(d) Show that an integral domain can be imbedded into a field. 10

Or

Find the field of quotients of the integral domain $\mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}\}$.

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2 0 1 2

MATHEMATICS

(Major)

Paper : 3.2

(Linear Algebra and Vector)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Linear Algebra)

(*Marks* : 40)

1. Answer the following : 1×6=6

(a) Is the following statement true or false? If false, correct the statement :

For linearly independent vectors v_1, v_2, v_3 in a vector space V the set $\{v_1, v_3\}$ is a linearly dependent set.

(b) What is the basis of the vector space $V = \{0_v\}$?

(c) State the condition under which a set of m vectors spans $\mathbb{R}^n$.

(d) What are the eigenvalues of an upper triangular matrix?

(e) State Cayley-Hamilton theorem.

(f) Let $T: V \to V$ be a linear operator. State one condition on T so that 0 is an eigenvalue of T.

2. Answer the following : 2×2=4

(a) Consider the vector space $V = \mathbb{R}^3$ over $\mathbb{R}$. If U and W are the xy-plane and yz-plane respectively, then determine $\dim(U \cap W)$.

(b) Find all eigenvalues of the operator

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$

defined by

$$T(x, y) = (3x + 3y, x + 5y)$$

3. Answer any *one* part : 10

(a) (i) Let V be the vector space of all functions from the real field $\mathbb{R}$ into $\mathbb{R}$. Show that W is a subspace of V, where

$$W = \{f : f(7) = f(1)\}$$

(ii) Let V be a finite d
space. Prove that
has the same nu

(iii) Determine wheth
following set S form

$S = \{(2, 4, -3), ($

(b) (i) Prove that the
finite collection o
vector space $V(F)$
$V(F)$. Is it true f
subspaces?

(ii) Let $V(F)$ be a
dimension n. Pro
vectors of V are li
Further prove that
$\cdots, v_n$ span V, then
independent.

(iii) Find a basis and
subspace U of $\mathbb{R}^4$,

$U = \{(a, b, c, d) \mid a$

4. Answer any *two* parts :

(a) (i) Let V be the vecto
square matrices o
and M be an arbit
Show that the
defined by $T(A) =$
is linear.

(ii) Let $T:\mathbb{R}^2 \to \mathbb{R}$ be the linear mapping for which $T(1, 1)=3$ and $T(0, 1)=-2$. Then find $T(x, y)$. 2+3=5

(b) Verify the rank nullity theorem for the linear mapping $T:\mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z)=(x+y, y+z)$$ 5

(c) Let U and V be vector spaces over a field K. If $\dim U=m$, $\dim V=n$, then prove that $\dim \operatorname{Hom}(U, V)=mn$, where $\operatorname{Hom}(U, V)$ denotes the vector space of all linear mappings from U into V. 5

5. Answer any *one* part : 10

(a) *(i)* Find the eigenvalues and the corresponding eigenvectors of the following matrix :

$$A=\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

(ii) Find the minimal polynomial of the matrix

$$A=\begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$$

Are the characteristic polynomial of A and the minimal polynomial of A same? 5+5=10

(b) *(i)* Show that the follo
linear equations
Hence solve them :

$x+2y-$
$3x-y+2$
$2x-2y+3$
$x-y+$

(ii) Prove that the mini
of a matrix A divi
nomial which has

(iii) Use Cayley-Hamilt
find the inverse of

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

GROUP—B

(Vector)

(*Marks* : 40)

6. Answer the following :

(a) Write the geometrical i
scalar triple product $\vec{a}\cdot$

(b) Does associative law for
of vectors hold?

(c) Write the condition for a vector function $\vec{f}$ of a scalar variable t to be of constant magnitude.

(d) Find div $\vec{r}$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

7. Answer the following : 2×3=6

(a) A particle moves along the curve $x = 3t^2$, $y = t^2 - 2t$, $z = t^3$, where t is the time. Find the component of velocity at time $t = 1$ in the direction $\hat{i} + \hat{j} - \hat{k}$.

(b) Show that the vector

$$\vec{v} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

is irrotational.

(c) Evaluate

$$\iint_S \vec{r} \cdot \hat{n} dS$$

where S is a closed surface.

(Symbols with usual meanings.)

8. Answer any *one* part : 10

(a) *(i)* Prove that

$$[\vec{b} \times \vec{c} \;\; \vec{c} \times \vec{a} \;\; \vec{a} \times \vec{b}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$$

(ii) Show that grad
perpendicular to
$\phi(x, y, z) = c$, where
Further show that c

(iii) Given

$$\vec{r}(t) = \hat{i} - 2\hat{j} + 2$$
$$= 2\hat{i} - \hat{j} + 4$$

then evaluate

$$\int_2^3 \left(\vec{r} \cdot \frac{d}{d} \right.$$

(b) *(i)* Prove that

$$[\vec{a} + \vec{b} \;\; \vec{b} + \vec{c} \;\; \vec{c} + \vec{c}$$

(ii) Find the angle betw
$x^2 + y^2 + z^2 = 9$ an
at the point (2, −1,

(iii) If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$, th

$$\int_C \vec{F} \cdot$$

where C is the curv
xy-plane from (0, 0)

9. Answer any *two* parts : 5×2=10

(a) *(i)* If

$$\frac{d\vec{r}}{dt} = \vec{w} \times \vec{r}$$

and

$$\frac{d\vec{s}}{dt} = \vec{w} \times \vec{s}$$

show that

$$\frac{d}{dt}(\vec{r} \times \vec{s}) = \vec{w} \times (\vec{r} \times \vec{s})$$

(ii) If

$$\vec{A} = \cos xy\hat{i} + (3xy - 2x^2)\hat{j} + (3x + 2y)\hat{k}$$

then find

$$\frac{\partial^2 A}{\partial x \partial y}$$

3+2=5

(b) If $\vec{a}$ is a constant vector and

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

show that—

(i) $\text{div}(\vec{a} \times \vec{r}) = 0$

(ii) $\text{curl}(\vec{a} \times \vec{r}) = 2\vec{a}$ 2+3=5

(c) Prove that

$$\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$ 5

(*Continued*)

10. Answer any *one* part :

(a) *(i)* Find the value of
vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i}$
$3\hat{i} + x\hat{j} + 5\hat{k}$ are copla

(ii) Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be thre
such that

$$\vec{a} \times (\vec{b} \times \vec{c}) =$$

If $\vec{b}$ and $\vec{c}$ are non-p
then find the angles
with $\vec{b}$ and $\vec{c}$.

(iii) If $\vec{F} = (2x^2 - 3z)\hat{i} - 2x$
evaluate

$$\int_V \text{div}\vec{F} \cdot dV$$

where V is the
bounded by the plan
$z = 0$ and $2x + 2y + z =$

(b) *(i)* Prove that

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\,\vec{b}\,\vec{d}]\vec{c}$$

$$= [$$

Hence express any
terms of $\vec{a}$, $\vec{b}$, $\vec{c}$ prov
not coplanar.

(ii) Evaluate

$$\iint_S \vec{F} \cdot \hat{n} dS$$

where

$$\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

and S is that part of the surface of the sphere $x^2 + y^2 + z^2 = 1$ which lies in the first octant. 5+5=10

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