Acc No — 35'18

2014

MATHEMATICS

(Major)

Paper : 3.1

(Abstract Algebra)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following as directed : $1\times10=10$

(a) Let G and G' be finite groups such that $\gcd(o(G), o(G'))=1$. Define a homomorphism from G to G'.

(b) State the fundamental theorem of group homomorphism.

(c) Let $f: G \to G'$ be a group homomorphism. Let $a \in G$ be such that $o(a)=n$ and $o(f(a))=m$. Then $o(f(a))/o(a)$ and f is one-one if and only if

(i) $m > n$

(ii) $m < n$

(iii) $m = n$

(iv) $m = n = 1$

(Choose the correct option)

(d) Let $R = \{0, 1, 2\} \bmod 3$. What is the characteristic of R?

(e) State whether True or False :

If an element a of a group G has only two conjugates in G, then $N(a)$ is a normal subgroup of G.

($N(a)$: normalizer of a in G)

(f) State Cauchy's theorem for a finite group G.

(g) If T is an automorphism of a group G, then $o(Ta) = o(a)$ for all $a \in G$. Now, for all $a, b \in G$

(i) $o(bab^{-1}) = o(b)$

(ii) $o(bab^{-1}) = o(a)$

(iii) $o(bab^{-1}) = o(Tb)$

(iv) $o(bab^{-1}) = 2$

(Choose the correct option)

(h) Give an example of a Euclidean domain.

(i) Let $R[x]$ be the
ring R and let
$f(x) = a$
and $g(x) = b$
If $f(x) + g(x) \neq 0$
(i) deg $(f(x) +$
(ii) deg $(f(x) +$
(iii) deg $(f(x) +$
(iv) deg $(f(x) +$
(C

(j) State True or
In a principa
non-zero prim

2. Answer the followin

(a) Let $\mathbb{Z}$ be the a
and $\phi : \mathbb{Z} \to \mathbb{Z}$
$x \in \mathbb{Z}$. Examine

(b) If R is a ring w
elements, the
idempotent e,

(c) The sum of tw
space is again
Justify whethe

(d) Let G be a group and $Z(G)$ be the centre of G. Show that if $\text{cl}(a) = \{a\}$, then $a \in Z(G)$. ($\text{cl}(a)$: the conjugacy class of a)

(e) Let f be a homomorphism from a ring R onto a ring R'. If e is the unity of R, then $f(e)$ is the unity of R'.

Justify whether this statement is true or false.

3. Answer the following questions : 5×4=20

(a) Let $G = (\mathbb{R}, +)$, $G' = (\{Z \in \mathbb{C} : |Z| = 1\}, \cdot)$ and $\phi : G \to G'$ is defined by

$$\phi(x) = \cos 2\pi x + i \sin 2\pi x, \quad x \in \mathbb{R}$$

Prove that ϕ is a homomorphism and determine $\ker \phi$.

Or

If $f : G \xrightarrow{\text{onto}} G'$ is a group homomorphism, prove that H is a normal subgroup of G if and only if $f(H)$ is a normal subgroup of G'.

(b) If R is a division ring, then show that the centre $Z(R)$ of R is a field.

Or

Let R be a ring having more than one element such that $aR = R$, for all $0 \neq a \in R$. Show that R is a division ring.

(c) Prove that a g
is prime, is A

(d) Prove that ev
domain is a

4. Answer the followi

(a) Let H and K
of a group G s

Let G be the
N be the su
integers. Prov
the group H
absolute valu

(b) Let A, B, C b
that $B \subseteq A$. S

$A \cap (B + C) =$

Let R be a co
Show that ev
also a prime
if every idea
is a field.

(c) State Sylow's 1st and 3rd theorems for a group G. Let $o(G) = pq$, where p, q are distinct primes such that $p < q$, $p \nmid q-1$. Show that G is cyclic. 2+8=10

Or

Let G be a finite group and $a \in G$. Prove that

$$o(\text{cl}(a)) = \frac{o(G)}{o(N(a))}$$

10

(d) Show that $\mathbb{Z}[\sqrt{2}] = \{a + \sqrt{2}b : a, b \in \mathbb{Z}\}$ is a Euclidean domain. 10

Or

Prove that any ring can be imbedded into a ring with unity.

★★★

2014

MATHEMATICS

(Major)

Paper : 3.2

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Linear Algebra)

(*Marks* : 40)

1. Answer the following as directed : 1×6=6

(a) Let W be a subset of the vector space $\mathbb{R}^3(\mathbb{R})$ defined by

$$W = \{(x, y, z) \mid x, y, z \in \mathbb{R} \text{ and } x + 2y + 4z = 4\}$$

Is W a subspace? Justify.

(b) Show that each non-zero singleton set $\{x\}$ of a vector space is linearly independent.

(c) If M is the vector space of all $m \times n$ matrices, determine dim M.

(d) The product of all the eigenvalues of a square matrix A is equal to

(i) 0 (ii) 1

(iii) $|A|$ (iv) $\frac{1}{|A|}$

(Choose the correct answer)

(e) The matrix

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$$

satisfies the equation

(i) $A^2 + 5A + 7I = 0$

(ii) $A^2 + 5A - 7I = 0$

(iii) $A^2 - 5A - 7I = 0$

(iv) $A^2 - 5A + 7I = 0$

(Choose the correct answer)

(f) State the condition for a system of n equations in n unknowns to have a unique solution.

2. Answer the following : 2×2=4

(a) Show that the union of two subspaces of a vector space is not necessarily a subspace of the vector space.

(b) Show that the matrices A and A' have the same eigenvalues.

3. Answer any *one* part :

(a) (i) Let U and V a vector spac that $V = U \oplus$ $U \cap W = \{0\}$.

(ii) Show that th (2, 5, 1) and independent i $\mathbb{R}^3(\mathbb{R})$.

(iii) If V is a finite space and { linearly indepe then prove tha to form a basi

(b) (i) If W_1 and W a finite dimen $V(F)$, then sho $\dim(W_1 + W_2) =$

(ii) Define basis Determine wh vectors (1, 1, 2 form a basis $\mathbb{R}^3(\mathbb{R})$.

4. Answer any *two* parts : 5×2=10

(a) Define rank and nullity of a linear transformation T from a vector space $V(F)$ to a vector space $W(F)$. Find rank and nullity of the linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (x+y+z, 2x+2y+2z)$$ 1+4=5

(b) Let T be a linear operator on a vector space $V(F)$ and rank T^2 = rank T. Then show that range $T \cap \ker T = \{0\}$. 5

(c) Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be a linear mapping defined by

$$T(x, y, z) = (3x+2y-4z, x-5y+3z)$$

Find the matrix A representing T relative to the bases $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ of $\mathbb{R}^3(\mathbb{R})$ and $B' = \{(1, 3), (2, 5)\}$ of $\mathbb{R}^2(\mathbb{R})$. 5

5. Answer any *one* part : 10

(a) (i) Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$$

(Continued)

(ii) State the Ca...
Verify it for

$A =$ ...

Hence find A^{-1}.

(b) (i) Let

$$A = \begin{pmatrix} 1 & 1 & \cdots \\ 0 & 2 & \cdots \\ 0 & 0 & \cdots \end{pmatrix}$$

Show that A...
characteristi...
the same m...

(ii) Show that th...

$3x$...
$-x$...
$2x+$...
$-2x+$...

is consiste...
them.

Group—B

(Vector)

(*Marks* : 40)

6. Answer the following as directed : 1×4=4

(a) Evaluate $(\vec{r}\cdot\hat{i})\hat{i}+(\vec{r}\cdot\hat{j})\hat{j}+(\vec{r}\cdot\hat{k})\hat{k}$.

(b) Write the value of $\operatorname{div}(\nabla\phi\times\nabla\psi)$.

(c) If ϕ is a continuously differentiable scalar point function, the value of curl grad ϕ is

(i) 1

(ii) −1

(iii) $\vec{0}$

(iv) None of the above

(Choose the correct answer)

(d) If C is a closed curve, find $\oint_C \vec{r}\cdot d\vec{r}$.

7. Answer the following : 2×3=6

(a) Prove the identity

$$\vec{a}\times[\vec{a}\times(\vec{a}\times\vec{b})]=(\vec{a}\cdot\vec{a})(\vec{b}\times\vec{a})$$

(b) Find div curl $\vec{F}$, if $\vec{F}=x^2y\hat{i}+xz\hat{j}+2yz\hat{k}$.

(c) Interpret the relations

$$\vec{r}\cdot\frac{d\vec{r}}{dt}=0 \text{ and } \vec{r}\times\frac{d\vec{r}}{dt}=\vec{0}$$

(*Continued*)

8. Answer any *one* p

(a) (i) Prove tha

$(\vec{a}\times\vec{b})\cdot(\vec{c}$

(ii) Prove

$\vec{r}=x\hat{i}+y$

(iii) If

$\vec{A}=$

and

$\vec{B}=$

calculate

at the po

(b) (i) Prove tha

$[\vec{b}+$

Hence s

coplanar

coplanar.

(ii) Prove that

$$\operatorname{curl}(f\vec{F}) = \operatorname{grad} f \times \vec{F} + f(\operatorname{curl}\vec{F})$$

where f is a scalar point function.

(iii) Find the unit normal vector to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$. 4+3+3=10

9. Answer any *two* parts : 5×2=10

(a) If $\vec{a}, \vec{b}, \vec{c}$ are three vectors, then prove that

$$[\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$$ 5

(b) Prove that

$$\operatorname{curl}(\vec{a}\times\vec{b}) = (\vec{b}\cdot\nabla)\vec{a} - (\vec{a}\cdot\nabla)\vec{b} + \vec{a}\operatorname{div}\vec{b} - \vec{b}\operatorname{div}\vec{a}$$ 5

(c) Show that

$$\int\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt = \vec{r}\times\frac{d\vec{r}}{dt} + C$$

where C is an arbitrary constant vector. If $\vec{r}(t) = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, then prove that

$$\int_1^2\left(\vec{r}\times\frac{d^2r}{dt^2}\right)dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$$

2+3=5

(*Continued*)

10. Answer any *one* p

(a) (i) Evaluate

a closed

cylinder

$x = 0,\ y =$

(ii) Verify Gre

for

$\oint_C (x^2$ -

where C is

(0, 0), (2,

(b) (i) Find the

a particle

the *xy*-pla

at the re

if the forc

$\vec{F} = (2x -$

(ii) State Gau

Using it,

$\vec{F} = x\hat{i} - y\hat{j}$

cylinder fo

$z = 0,\ z = 1,$