2015

MATHEMATICS

(Major)

Paper : 3.1

(Abstract Algebra)

Full Marks – 80

Time – Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed : 1×10=10

(a) Let G be a group and $f : G \to G$ such that $f(x) = x^{-1}$ be a homomorphism.

Then G is abelian.

–Justify whether it is true or false.

(b) Let G and G' be two groups and $\phi : G \to G'$ be a homomorphism. If $a \in G$ and o(a) is finite then examine whether $o(\phi(a))$ is a divisor of o(a).

(c) Let G be a group and N be a normal subgroup of G. Define the canonical homomorphism from G to the quotient group G/N.

(d) Choose the correct option :

If the characteristic of an integral domain D is a non-zero number p, then the order of every non-zero element in the group (D, +) is

(i) $p+1$ (ii) p

(iii) $p-1$ (iv) none of these

(e) If G is an infinite cyclic group, then Aut (G) (the set of all automorphisms on G) is a group of order 2.

–State whether true or false.

(f) Define inner automorphism of a group G.

(g) If K is the only Sylow p-subgroup of a group G, then K is normal in G.

–Justify whether it is true or false.

(h) Define a ring homo
complex numbers.

(i) Give an example
domain.

(j) Let I be an ideal
commutative, the
commutative.
–State whether it

2. Answer the following

(a) Let G be a group
a group of order 6.
exist a homomorph

(b) If a is an invertible
unity, then show th
zero.

(c) If U and W are tw
space V, then show
subspace of V con
and W.

(d) Show that a group
a prime, may not

(e) Let R be a ring with unity 1 and f is a homomorphism of R into an integral domain R′. If Ker f ≠ R, prove that f(1) is the unity of R′.

3. Answer the following questions : 5×4=20

(a) Let G and G′ be two groups and ϕ be a homomorphism from G onto G′. Prove that if G is commutative, then G′ is also commutative, and if G is cyclic then G′ is also cyclic.

Or

Show that any infinite cyclic group is isomorphic to the additive group of integers, and any finite cyclic group of order n is isomorphic to $\mathbb{Z}_n$, the group of integers modulo n.

(b) If in a ring R with unity $(xy)^2 = x^2y^2$, for all x, y ∈ R, then show that R is commutative.

Or

Let R be a finite (non-zero) integral domain. Then prove that $o(R) = p^n$, where p is a prime.

(c) Let G be a group.
inner automorphism
the group of all a

(d) Let R[x] be the r
a ring R. Show that
only if R[x] is co

4. Answer the following

(a) Prove that the set A
of S_n $(n \geq 2)$ is a
and $o(A_n) = \frac{1}{2}$ o(
subgroups of S_4.

Or

Let f be a homom
onto a group G′. L
and H′ be a subgr
Prove that

(i) f(H) is a subg

(ii) $f^{-1}(H')$ is a su
K = Ker f.

(iii) there exists a one-to-one correspondence between the set of subgroups of G containing K and the set of subgroups of G′. 3+3+4=10

(b) Let R be a commutative ring. Prove that an ideal P of R is prime if and only if R/P is an integral domain. Moreover, if R is with unity and M is a maximal ideal of R such that $M^2 = \{o\}$, then show that for any other maximal ideal N of R, N = M. 6+4=10

Or

Show that the union of two subspaces of a vector space may not be a subspace. Consider the vector space $V(F) = F^2(F)$, where F is a field. Let $W_1 = \{(a, o) : a \in F\}$ and $W_2 = \{(o, b) : b \in F\}$ Show that $V = W_1 \oplus W_2$. 4+6=10

(c) Let G be a finite group and x, y be conjugate elements of G. Show that the number of distinct elements $g \in G$ such that $g^{-1}xg = y$ is $o(N(x))$. 10

Or

Prove that the num
conjugacy class c(a
order is $o(G)/o(N($
normalizer of a. If
of G, then prove th

$o(G) = o(Z(G)) + _{a}$

(d) If D_1 and D_2 are
domains, then sho
fields of quotients
isomorphic.

Or

Show that an int
imbedded into a fie

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3 (Sem 3) MAT M2

2015

MATHEMATICS

(Major)

Paper : 3.2

(Linear Algebra and Vector)

Full Marks – 80

Time – Three hours

The figures in the margin indicate full marks for the questions.

GROUP – A

(Linear Algebra)

Marks : 40

1. Answer the following as directed : 1×7=7

(a) Show that in a vector space V(F)

$\alpha v = 0,\ v \neq 0 \Rightarrow \alpha = 0$

$\alpha v = 0,\ \alpha \neq 0 \Rightarrow v = 0$, where $v \in V$, $\alpha \in F$.

[Turn over

(b) Let S be the subset of $\mathbb{R}^3$ defined by

$$S=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=z^2\}$$

Examine whether S is a subspace of $\mathbb{R}^3$.

(c) In $\mathbb{R}^3$, $\alpha = (4, 3, 5)$, $\beta = (0, 1, 3)$, $\gamma = (2, 1, 1)$.

Is α a linear combination of β and γ ?

(d) Let $T:\mathbb{R}^2\to\mathbb{R}^3$ be defined by

$T(x_1,x_2)=(x_1,x_1+x_2,x_2)$

Examine whether T is a linear transformation.

(e) Let T be a linear operator on $\mathbb{R}^2$ which is represented by the following matrix :

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with respect to the standard ordered basis. Then T has no eigen value in $\mathbb{R}$. –Justify whether it is true or false.

(f) Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be defined by $T(x,y)=(x,0)$. What is the eigen space of T associated with the eigen value 1 ?

(g) If λ is a simple e
plicity 1) of an n×
$(A-\lambda T)$ is

(i) n+1

(ii) n–1

(iii) n

(iv) none of these

—Choose the cor

2. Answer the following

(a) Let S and T be tw
of a vector space
Show that L(S) ⊂L
denote the linear spa

(b) Let v and u be ve
field F and T
transformation. Sh
only if T is one-c

(c) A linear transforma
by $T(x,y,z)=(3x$

Find the matrix o
bases $\{(1,0,0),(0,$
$\{(1,0),(0,1)\}$ of

(d) Using Cayley-Hamilton theorem, compute the inverse of $A=\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$.

3. Answer any *one* part : 5

(a) Find the range, rank, kernel and nullity of the following linear transformation :

$T:\mathbb{R}^2 \to \mathbb{R}^3$ such that
$T(x,y)=(x+y, x-y, y)$.

(b) Determine the linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^2$ which maps the basis vectors (1, 0, 0), (0, 1, 0), (0, 0, 1) of $\mathbb{R}^3$ to the vectors (1, 1), (2, 3) and (3, 2) respectively. Also determine ker T.

4. Answer the following questions : 10×2=20

(a) Let W be a subspace of a finite dimensional vector space V. Then show that W is also finite dimensional and $\dim W \le \dim V$. Also show that $\dim V = \dim W$ if and only if $V = W$.

Or

Let W be a subspace of a finite dimensional vector space V. Prove that there exists a subspace W' of V such that $V = W \oplus W'$.

(b) Prove that simil
characteristic poly
matrix. Let λ be a r
that there exists
corresponding to e
also real.

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Obtain the eigen
eigen spaces of th

A =

Is A diagonalisabl

GROU

(Ve

Marks

5. Answer the following

(a) If $\vec{a}=-3\hat{i}+7\hat{j}+$
$\vec{b}=-5\hat{i}+7\hat{j}-$
$\vec{c}=7\hat{i}-5\hat{j}-3$

Find $\vec{a}\times(\vec{b}\times\vec{c})$.

(b) Examine whether $\vec{a}-2\vec{b}+3\vec{c})$, $-2\vec{a}+3\vec{b}-4\vec{c})$ and $\vec{a}-3\vec{b}+5\vec{c})$ are coplanar.

(c) $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})$ is equal to

(i) $2\vec{b}$

(ii) $2\vec{a}$

(iii) 0

(iv) none of these

— Choose the correct option.

6. Show that if $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$, then

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right]^2=\vec{a}^2\left(\vec{b}\times\vec{c}\right)^2$$ 2

7. Answer the following questions : 5×3=15

(a) Show that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\times\vec{c}$ if and only if either $\vec{b}=0$, or $\vec{c}$ is collinear with $\vec{a}$, or $\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{c}$.

Or

If $\vec{a}=(1,1,1)$, $\vec{b}=(2,-1,3)$, $\vec{c}=(1,-1,0)$, $\vec{d}=(6,2,3)$, express $\vec{d}$ in terms of $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$.

(b) If $\vec{r}=a\cos t\,\hat{i}+a\sin t$

find $\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right|$

and $\left[\frac{d\vec{r}}{dt}\;\frac{d^2\vec{r}}{dt^2}\;\frac{d^3\vec{r}}{dt^3}\right]$

(c) Determine the consta

$\vec{f}=(x+3y)\hat{i}+(y-2z$

solenoidal.

8. Answer the following qu

(a) A particle moves so

is given by $\vec{r}=\cos\omega$

a constant. Show tha

(i) the velocity of th

to $\vec{r}$

(ii) the acceleration

origin and has m

the distance fro

(iii) $\vec{r}\times\frac{d\vec{r}}{dt}$ is a co

Or

If $\vec{a}$ is a constant vector, prove that

$$\text{div}\left(r^n(\vec{a}\times\vec{r})\right)=0,$$

where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$. 10

(b) Evaluate : $\iint_S (y^2z^2\hat{i}+z^2x^2\hat{j}+x^2y^2\hat{k}).d\vec{S}$,

where S is the part of the sphere

$x^2+y^2+z^2=1$ above the xy-plane. 10

Or

Find the work done when a force $\vec{F}=(x^2-y^2+x)\hat{i}-(2xy+y)\hat{j}$ moves a particle in xy-plane from (0, 0) to (1, 1) along the parabola $y^2=x$. If C is the circle $x^2+y^2=4, z=0$, find the circulation of $\vec{F}$ along the curve C.

5+5=10