2019

MATHEMATICS

(Major)

Paper : 4.1

(Real Analysis)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following as directed : 1×10=10

(a) Let S be a nonempty subset of R that is bounded below. Then choose the correct option for

$$k = \sup S / \inf S / -\sup S / -\inf S$$

if $k = -\sup\{-s \in S\}$.

(b) If $G_n = (0, 1 + 1/n)$ for $n \in N$, then the intersection $\bigcap_{n=1}^{\infty} G_n$ is open.

(Write True or False)

(c) For $\{x_n\}$ given by the formula $x_n = n/(n+1)$, establish either the convergence or the divergence of the sequence $\{x_n\}$.

(d) If β a limit point of a sequence $\{S_n\}$, then there exists a subsequence $\{S_{n_k}\}$ of $\{S_n\}$ which converges to β.

(Write True or False)

(e) If Σu_n is a positive term series such that $\lim_{n \to \infty} (u_n)^{1/n} = L$, then under what condition the Cauchy's root test confirms divergence of Σu_n?

(f) The series

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$$

is not convergent. Give reason.

(g) Define limit of a function (sequential approach).

(h) Is the function f, where $f(x) = \frac{x - |x|}{x}$ continuous?

(i) If the function defined on the closed interval $[a, b]$ satisfies the conditions of the mean-value theorem and $f'(x) = 0$ for all $x \in]a, b[$, then verify that $f(x)$ is constant on $[a, b]$.

(j) A function f is defined on R by

$$f(x) = \begin{cases} x, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x \ge 1 \end{cases}$$

Is $f'(1)$ exist? Justify.

2. Answer the following questions : $2\times5=10$

(a) Any open interval $I = (a, b)$ is an open set. Why?

(b) Show that the series $\sum \frac{1}{n}$ does not converge.

(c) Prove that if f is continuous on $[a, b]$ and $f(x) \in [a, b]$ for every $x \in [a, b]$, then there exists a point $c \in [a, b]$ such that $f(c) = c$.

(d) Show that the maximum value of $(\log x) / x$ in $0 < x < \infty$ is $1/e$.

(e) Show that the function $f(x) = x^{1/3}$, $x \in R$, is not differentiable at $x = 0$.

3. Answer any *four* parts : $5\times4=20$

(a) Prove that the intersection of any finite number of open sets is open. Does this result hold for arbitrary family of open sets? Justify it.

(b) If $\{a_n\}$ and $\{b_n\}$ be two sequences such that $\lim a_n = a$ and $\lim b_n = b$, then prove that $\lim (a_n b_n) = ab$.

(c) Prove that a positive term series Σu_n, where $u_n = \frac{1}{n^p}$, is convergent if $p > 1$.

(d) Test for convergence of the series

$$\sum \frac{(n!)}{(2n)!} x^n, \; x > 0$$

(e) Prove that a function f, which is continuous on a closed interval $[a, b]$, assumes every value between its bounds.

(f) Expand, if possible, the function $f(x) = \sin x$ in ascending powers of x.

4. Answer *either* *(a)* and *(b)* *or* *(c)* and *(d)* : 5×2=10

(a) State and prove Bolzano-Weierstrass theorem for sets. 1+4=5

(b) Show that the sequence $\{a_n\}$, where

$$a_n = \left\{ \frac{1}{\sqrt{(n^2+1)}} + \frac{1}{\sqrt{(n^2+2)}} + \cdots + \frac{1}{\sqrt{(n^2+n)}} \right\}$$

converges to 1. 5

(c) If $\{b_n\}$ be a sequence of positive real numbers such that $b_n = \sqrt{b_{n-1}b_{n-2}}$, $n > 2$, then show that the sequence converges to $(b_1 b_2^2)^{1/3}$. 5

(d) Prove that a monotonic increasing bounded above sequence converges to its least upper bound. 5

5. Answer *either* (a) and (b) *or* (c) and (d) : 5×2=10

(a) State Abel's test and show that

$$1 - \frac{1}{3\cdot 2^2} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 4^2} + \cdots$$

is convergent. 1+4=5

(b) Test for convergence of the following series whose nth term is given by

$$\frac{1.3.5\cdots(4n-3)}{2.4.6\cdots(4n-2)}\cdot\frac{x^{2n}}{4n}$$ 5

(c) State the comparison test (limit form) and using it, test the convergence of the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}$. 1+4=5

(d) When an infinite series Σu_n is said to be absolutely convergent? Prove that if Σu_n absolutely convergent, then $\Sigma |u_n|$ is convergent. Does the divergence of $\Sigma |u_n|$ imply the divergence of Σu_n? 1+3+1=5

6. Answer any *two* parts : 5×2=10

(a) Show that the function f defined on R by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point. 5

(b) Evaluate : 2+3=5

(i) $\lim_{x \to 0} \frac{e^{1/x}}{e^{1/x}+1}$

(ii) $\lim_{x \to 0} \frac{1-2\cos x+\cos 2x}{x^2}$

(c) Prove that a continuous and strictly increasing function f in $[a, b]$ is invertible and the inverse function is continuous in $[f(a), f(b)]$. 5

(d) Show that

$$\frac{\tan x}{x} > \frac{x}{\sin x} \text{ for } 0 < x < \frac{\pi}{2}$$ 5

7. Answer any *two* parts : 5×2=10

(a) If f is derivable at c and $f(c) \neq 0$, then prove that the function $\frac{1}{f}$ is also derivable thereat, and then obtain the result

$$\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{\{f(c)\}^2}$$ 5

(b) State and prove Roll's theorem. 1+4=5

(c) If c is an interior point of the domain of a function f and $f'(c) = 0$, then show that the function has a maxima or a minima at c, according as $f''(c)$ is negative or positive. 5

(d) Use Taylor's theorem with $n = 2$ to approximate $\sqrt[3]{1+x}$, $x > -1$. 5

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2019

MATHEMATICS

(Major)

Paper : 4.2

(Mechanics)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : 1×10=10

(a) What is the length of arm of the couple whose constituent force is of magnitude F and equivalent to the couple (P, p)?

(b) Define moment of a force about a line.

(c) What is the position of CG of an uniform triangular lamina?

(d) Explain briefly why the force of friction is called a passive force.

(e) Write one characteristic of the central axis of a system of forces.

(f) A point is moving in a straight line with SHM about a fixed point O of the line. If μ be the intensity and a be the amplitude of the motion; write down the expression for velocity at a distance x from O.

(g) Write down the relation between linear velocity and angular velocity of a particle moving in a plane curve.

(h) Explain why the virtual work done by reaction R of any smooth surface is neglected in forming equation of virtual work.

(i) Define central force and give one example.

(j) Write down the differential equation of the path of a particle moving under a central force, in pedal form.

2. Answer the following questions : $2\times5=10$

(a) A body of weight 4 lb rests in limiting equilibrium on an inclined plane whose slope is 30°. Find the coefficient of friction and the normal reaction.

(*b*) Show that, if the displacement of a particle moving in a straight line is expressed by the equation

$$x = a\cos nt + b\sin nt$$

it describes a simple harmonic motion.

(*c*) Explain the dynamical significance of Kepler's 3rd law of motion.

(*d*) Two forces P and Q, one acts along $y = 0$, $z = 0$ and the other along $x = 0$, $z = c$. Find the components of couple L, M, N.

(*e*) A particle is describing an ellipse $l/r = 1 + e\cos\theta$ under a force to a pole. Find the law of force.

3. Answer any *four* of the following questions : 5×4=20

(*a*) The algebraic sum of the moments of a system of coplanar forces, about points whose coordinates are (1, 0), (0, 2) and (2, 3) referred to rectangular axes are G_1, G_2 and G_3 respectively. Find the tangent of the angle which the direction of the resultant force makes with the axis of x.

(b) A particle is moving in a plane curve. Obtain the expressions for acceleration of the particle along and perpendicular to the radius vector.

(c) Establish the statement :

A system of forces acting on a rigid body can be reduced to a single force and a couple whose axis is along the direction of the force. Hence define Poinsot's central axis.

(d) A particle moves in a straight line from a distance a towards the center of force, the force varying inversely as the cube of the distance. Show that the time to descent to the centre is $a^2 / \sqrt{\mu}$.

(e) Find the position of CG of the arc of the cardioide $r = a(1 + \cos\theta)$ lying above the initial line.

(f) If ω be the angular velocity of a planet at the nearer end of the major axis, prove that its period is

$$-\frac{2\pi}{\omega}\sqrt{\frac{1+e}{(1-e)^3}}$$

4. (a) Prove that a force acting at any point of a body is equivalent to an equal and parallel force acting at any other arbitrary point of the body together with a couple. 2

(b) P and Q are two like parallel forces. If a couple, each of whose forces is F and whose arm is a in the plane of P and Q, is combined with them, then show that the resultant is displaced through a distance $\frac{Fa}{P+Q}$. 4

(c) A beam whose centre of gravity divides it into two portions a and b is placed inside a smooth sphere. Show that if θ be its inclination to the horizon in the position of equilibrium, and 2α be the angle subtended by the beam at the centre of the sphere, then

$$\tan\theta = \frac{b-a}{b+a}\tan\alpha$$ 4

5. Answer *either* (a) *or* (b) :

(a) (i) Find the CG of a thin uniform hemispherical shell. 4

(ii) Four rods of equal weight w form a rhombus $ABCD$, with smooth hinges at the points. The frame is suspended by the point A, and a weight W is attached at C. A stiffening rod of negligible weight joins the middle points of AB and AD, keeping these inclined at an angle α to AC. Show that the thrust in this stiffening rod is

$$(2W + 4w)\tan\alpha$$ 6

(b) (i) Find the centroid of the area included between the curve $y^2 = x$ and the straight line $y = x$. 5

(ii) A square of side $2a$ is placed with its plane vertical between two smooth pegs which are in the same horizontal line and at a distance c. Show that it will be in equilibrium when the inclination of one of its edges to the horizon is either

$$\frac{\pi}{4} \text{ or } \frac{1}{2}\sin^{-1}\frac{a^2 - c^2}{c^2}$$ 5

6. (a) Define stable and unstable equilibrium of a body. Explain how to determine nature of stability of equilibrium of a body having one degree freedom. 2+3=5

(b) A lamina in the form of an isosceles triangle whose vertical angle is α, is placed on a sphere of radius r so that its plane is vertical and one of the equal sides is in contact with the sphere. Show that if the triangle be slightly displaced in its own plane, the equilibrium is stable if

$$\sin\alpha < \frac{3r}{a}$$

where a is one of the equal sides. 5

Or

A sphere of weight W and radius a lies within a fixed spherical shell of radius b and a particle of weight w is fixed to the upper end of the vertical diameter. Prove that equilibrium is stable if

$$\frac{W}{w} > \frac{b-2a}{a}$$

7. Answer any *two* of the following questions : 5×2=10

(a) A spherical raindrop, falling freely receives in each instant an increase of volume equal to λ times its surface at that instant. Find the velocity at the end of time t.

(b) A particle falls under gravity (supposed constant), from rest in a medium whose resistance varies as the square of the velocity. Show that in time t it has fallen through a distance

$$x = \frac{V^2}{g} \log \cosh \frac{gt}{V}$$

V being the terminal velocity.

(c) A particle is projected along the inner surface of a rough sphere and is acted on by no forces. Show that it will return to the point of projection after time

$$\frac{a}{\mu V}(e^{2\pi\mu} - 1)$$

where a is the radius of the sphere and V is the velocity of projection.

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