2019

MATHEMATICS

(Honours)

Paper : MAT–HC–1026

(Algebra)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : 1×10=10

(a) Find the polar coordinates of the point $(6, 6\sqrt{3})$.

(b) For $z_1, z_2 \in C$, is the number $z_1\bar{z}_2 + \bar{z}_1 \cdot z_2$ a real number?

(c) Using quantifiers, write the statement "In this book some pages do not contain any picture."

(d) Is the function $f : Z \to Z$ defined by $f(x) = 3x + 7$ one-one?

(e) Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$. Consider the subset R of $X \times Y$ as $R = \{(a, 1), (a, 3), (b, 2)\}$. Is there any element in X, which is not related to any element in Y under R? Justify.

(f) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = mx + b$. Under what condition f is linear?

(g) Write the system as a vector equation and then as a matrix equation :

$$\begin{aligned} 8x_1 - x_2 &= 4 \\ 5x_1 + 4x_2 &= 1 \\ x_1 - 3x_2 &= 2 \end{aligned}$$

(h) If $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$, then find $\det(A - B)$.

(i) Is $\mathbb{N}$ and $2\mathbb{N}$, the set of even positive integers, have the same cardinality?

(j) Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : x = 1\}$. Find $A \cap B$.

2. Answer the following questions : 2×5=10

(a) Find the geometric image of the complex number z in $|z - 2| = 3$.

(b) For what values of h and k, the following system of equations is consistent?

$$2x_1 - x_2 = h$$
$$-6x_1 + 3x_2 = k$$

(c) Write the negation of the following statements :

(i) $A : \exists\, x \in X$ (x has property P and Q)

(ii) $B : \forall\, x \in X$ (x has property P or Q)

(d) Find the fourth roots of unity and interpret the result geometrically.

(e) Consider the relation on $\mathbb{R}$ with the defining set

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : xy > 0\} \cup \{(0, 0)\}$$

Is $\mathbb{R}$ an equivalence relation?

3. Answer any *four* questions of the following : 5×4=20

(a) Find the polar representation of the complex number

$$z = 1 + \cos\alpha + i\sin\alpha,\ \alpha \in (0, 2\pi)$$ 5

(b) Prove that for any sets A, B and C

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$ 5

(c) Let $f : X \to Y$ be a map and $B_1, B_2 \subseteq Y$. Prove that

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$ 5

(d) Balance the chemical equation using the vector equation approach. When solutions of sodium phosphate and barium nitrate are mixed, the result is barium phosphate and sodium nitrate. The unbalanced equation is

$$Na_3PO_4 + Ba(NO_3)_2 \to Ba_3(PO_4)_2 + NaNO_3$$ 5

(e) Let $T(x, y) = (3x + y, 5x + 7y, x + 3y)$. Show that T is a one-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$? 4+1=5

(f) What is the correspondence between the linear independence of the columns of a matrix A and the equation $A\vec{x} = \vec{0}$? Use this fact to check the columns of matrix given below are a linearly independent set :

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 1 & 7 \end{bmatrix}$$ 1+4=5

4. Answer any *four* questions of the following : 10×4=40

(a) *(i)* Compute : 5

$$z = \frac{(1-i)^{10}(\sqrt{3}+i)^5}{(-1-i\sqrt{3})^{10}}$$

(ii) Let z_1, z_2, z_3 be complex numbers, such that $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$. Then prove that

$$z_1^3 + z_2^3 + z_3^3 = 0$$ 5

(b) *(i)* If $f : X \to Y$ and $g : Y \to X$ be such that $g \circ f = I_{dX}$ and $f \circ g = I_{dY}$, then prove that f and g are bijective. 5

(ii) Let $f : \mathbb{R} \to [0, \infty)$ be defined by $f(x) = x^2$ and $g : [0, \infty) \to \mathbb{R}$ defined by $g(x) = \sqrt{x}$, the unique non-negative square root of x. Find $f \circ g$ and $g \circ f$. Is $f \circ g = g \circ f$? If not, when are they equal? 5

(c) Define equivalence relation on a non-empty set X. Show that the relation congruence modulo n, where $n \neq 0$, is any fixed integer on the set Z of integers, defined by

$$a \equiv b \pmod{n} \text{ iff } n / a - b$$

is an equivalence relation. Find all the distinct equivalence classes of Z if $n = 4$, so that Z is the union of these. 1+4+5=10

(d) Define well-ordering principle. Prove that if $a, b \in Z$ with $a \in \mathbb{N}$, then there exists unique integers q and r such that—

(i) $b = aq + r$;

(ii) $0 \leq r < a$. 2+8=10

(e) *(i)* Write true or false and give their converse statements :

A : If the apple is red, then it is ripe.

B : In case the bakery is open, I will buy a cake for you. 4

(ii) Write the contrapositive and negation of the following statements :

P : If the boy owns a BMW car, then he is rich.

Q : For an integer n, if $n^2 < 20$, then $n < 5$.

R : For an integer x, if $x^2 - 6x + 5$ is even, then x is odd. 6

(f) Define a homogeneous system of linear equations. Write the solution set of the given homogeneous system in parametric vector form :

$$x + 3y - 5z = 0$$
$$x + 4y - 8z = 0$$
$$-3x - 7y + 9z = 0$$

Also describe the solution set of the following system in parametric vector form :

$$x + 3y - 5z = 4$$
$$x + 4y - 8z = 7$$
$$-3x - 7y + 9z = -6$$

Provide a geometric comparison between the two solution sets.

2+3+3+2=10

(g) *(i)* If A is an $n \times n$ invertible matrix, then for each $\overline{b}$ in $\mathbb{R}^n$, prove that the equation $A\overline{x} = \overline{b}$ has a unique solution $\overline{x} = A^{-1}\overline{b}$. 2

(ii) Prove that an $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n. Also, any sequence of elementary operations that reduces A to I_n also transforms I_n to A. 3

(iii) Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

if it exists by performing suitable row operations on the augmented matrix $[A : I]$. 5

(h) *(i)* Prove that an index set $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors is a linear combination of the others. 4

(ii) Find the area of the parallelogram whose vertices are $(-1, 0)$, $(0, 5)$, $(1, -4)$ and $(2, 1)$ using determinant. 3

(iii) If A and B are $n \times n$ matrices, then prove that

$$\det(AB) = (\det A)(\det B)$$ 3

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