2018

MATHEMATICS

(Major)

Paper : 5.1

(Real and Complex Analysis)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : 1×7=7

(a) Evaluate :

$$\lim_{(x, y)\to(0, 0)} \frac{x^2y^2}{x^2y^2+(x^2-y^2)^2}$$

(b) Find the infimum of all upper sums of the function $f(x)=3x+1$ on the interval [1, 2].

(c) When is an improper integral said to be convergent?

(d) Define uniform continuity of a function whose domain and codomain are set of complex numbers.

(e) Justify whether true of false :

"If a complex valued function $f(z)$ is analytic, then the real part of $f(z)$ is harmonic."

(f) Verify whether the transformation $w = z^3$ is conformal or not at all points of the region $|z| < 1$.

(g) Write the physical effect of a region transformed from z-plane to w-plane under the transformation $w = az + b$; a, b are given complex constants.

2. Answer the following questions : 2×4=8

(a) Show that the following function is discontinuous at the origin :

$$f(x, y) = \begin{cases} \dfrac{1}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

(b) Prove that for a bounded function f

$$\underline{\int_a^b} f\,dx \le \overline{\int_a^b} f\,dx$$

(Symbols have their usual meaning.)

(c) Test the convergence of

$$\int_0^1 \frac{dx}{\sqrt{1-x^3}}$$

(d) Prove that the cross ratio is an invariant quantity under bilinear transformation.

3. Answer any *three* parts : 5×3=15

(a) Prove that if f_x and f_y are both differentiable at a point (a, b) of the domain of definition of a function f, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

(Symbols have their usual meaning.)

(b) Prove that a monotonic function on a closed interval is integrable therein.

(c) Show that the integral

$$\int_0^{\pi/2} \frac{\sin^m x}{x^n} dx$$

exists, iff $n < m+1$.

(d) Let $f(z) = u + iv$, z is a complex number, be analytic in a region R. Prove that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

(e) Let $f(z)$, z is a complex number, be analytic inside and on the boundary C of a simply connected region R. Prove that

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2} dz$$

4. Answer *either* (a) *or* (b) :

(a) (i) Prove that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$$

is invariant for change of rectangular axes. 5

(ii) If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, then show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$$ 5

(b) (i) Show that the function f defined as

$$f(x) = \frac{1}{2^n}$$

when $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$, $f(0) = 0$ is integrable on $[0, 1]$. 5

(ii) If f and g are both differentiable on $[a, b]$ and if f' and g' are both integrable on $[a, b]$, then prove that

$$\int_a^b f(x)\,g'(x)\,dx = [f(x)\ g(x)]_a^b - \int_a^b g(x)\,f'(x)\,dx$$ 5

5. Answer *either (a) or (b)* :

(a) (i) Prove that if f is bounded and integrable on $[a, b]$, then $|f|$ is also bounded and integrable on $[a, b]$ but the converse is not true. 5

(ii) Find a bilinear transformation that maps points $z = 0, -i, -1$ into $w = i, 1, 0$, respectively. 5

(b) (i) For what value of m and n is the integral

$$\int_0^1 x^{m-1}(1-x)^{n-1} \log x \, dx$$

convergent? 5

(ii) Show that if f and g are positive in $[a, x]$ and

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = l$$

where l is a non-zero finite number, then the two integrals

$$\int_A^\infty f \, dx \text{ and } \int_a^\infty g \, dx$$

converge or diverge together. 5

6. Answer *either* (a) *or* (b) :

(a) (i) Prove that $f(z) = z^3$ is uniformly continuous but

$$f(z) = \frac{1}{z^3}$$

is not uniformly continuous in the region $|z| < 1$. 5

(ii) Find a function v such that $f(z) = u + iv$, z is a complex number, is analytic, where

$$u = x^2 - y^2 - 2xy - 2x + 3y$$ 5

(b) *(i)* Evaluate $\int_C \bar{z}\, dz$ along the curve C given by the line from $z = 0$ to $z = 3i$ and then the line from $z = 3i$ to $z = 6 + 3i$. 5

(ii) Evaluate

$$\oint_C \frac{z^2}{(z-1)(z-2)} dz$$

and $\oint_C \frac{e^{2z}}{(z+1)^4} dz$, where C is the circle $|z| = 3$. 3+2=5

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