2018

MATHEMATICS

(Major)

Paper : 5.2

(Topology)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : 1×7=7

(a) Describe the open spheres for any discrete metric space (X, d).

(b) Find the derived sets of the following subsets of $\mathbb{R}$:

$$A =]0,\ 1], \quad B = \left\{\frac{2n+1}{n} : n \in \mathbb{N}\right\}$$

$$C = \left\{-\frac{1}{n} : n \in \mathbb{N}\right\}$$

(c) Define a Cauchy sequence in a metric space (X, d).

(d) Define a topological space and give one example.

(e) Let

$X = \{a, b, c\}$ and

$\mathscr{T} = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$

is a topology on X. Find the derived set of $A = \{a, b\}$.

(f) Let $\mathscr{T}$ be the topology on $\mathbb{N}$ which consists of ϕ and all subsets of the form $G_m = \{m, m+1, m+2, \ldots\}$, $m \in \mathbb{N}$. What are the open sets containing 4?

(g) What do you mean by a Banach space? Give one example.

2. Answer the following questions : 2×4=8

(a) Show that every closed interval is a closed set in the usual metric on $\mathbb{R}$.

(b) Let f be a mapping from $\mathbb{R}$ into $\mathbb{R}$ defined by

$$f(x) = \begin{cases} -2 & \text{when } x < 0 \\ 2 & \text{when } x \geq 0 \end{cases}$$

Examine whether f is continuous with respect to the usual topology on $\mathbb{R}$.

(c) Let $(X, \|\cdot\|)$ be a normed linear space and $x_n \to x$ and $y_n \to y$ in X. Show that $x_n + y_n \to x + y$.

(d) Prove the parallelogram law in an inner product space $(X, \langle \cdot, \cdot \rangle)$.

3. Answer the following questions : 5×3=15

(a) Let (X, d) be a metric space and A and B be subsets of X. Prove that—

(i) $A \subset B \Rightarrow D(A) \subset D(B)$

(ii) $D(A \cup B) = D(A) \cup D(B)$

(b) Let X be any set and $\mathscr{T}$ be the collection of all those subsets of X whose complements are finite together with the empty set. Show that $\mathscr{T}$ is a topology on X. What do you call this topology?

Or

Let $(X, \mathscr{T})$ be a topological space and $A \subset X$. Prove that $\overline{A} = A \cup D(A)$.

(c) Show that $\mathbb{R}^n$ is a normed linear space with some suitable norm.

Or

Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. Prove that for all $x, y \in X$

$$4\langle x, y\rangle = ||x+y||^2 - ||x-y||^2 + i||x+iy||^2 - i||x-iy||^2$$

4. Answer the following questions : 10×3=30

(a) Prove that every non-empty open set on the real line is the union of a countable class of pairwise disjoint open intervals.

Or

State and prove Cantor's intersection theorem for metric spaces.

(b) Let (X, d) be a metric space and $x_0 \in X$ be fixed. Show that the real-valued function $f_{x_0}(x) = d(x, x_0)$, $x \in X$ is continuous. Is it uniformly continuous? Let (Y, P) be another metric space and $f : X \to Y$ be a mapping. Prove that f is continuous if and only if the inverse image of every open set in Y is an open set in X. 2+1+7

Or

Let X be a metric space and Y be a complete metric space. Let A be a dense subspace of X. If $f : A \to Y$ is uniformly continuous, then prove that f can be extended uniquely to a uniformly continuous mapping $g : X \to Y$.

(c) Prove that a metric space is compact if and only if it is complete and totally bounded.

Or

Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of connected subsets of a space X such that

$$\bigcap_{\lambda \in \Lambda} A_\lambda \neq \phi$$

Prove that $\bigcup_{\lambda \in \Lambda} A_\lambda$ is a connected set in X.

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