2017

MATHEMATICS

(Major)

Paper : 5.1

(Real and Complex Analysis)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : 1×7=7

   (a) Define limit of a function of two variables.

   (b) Write a sufficient condition for the equality of f_{xy} and f_{yx}, symbols have their usual meanings.

   (c) Give an example of a function which is not continuous but Riemann integrable.

(d) Find the fixed points of the transformation $w = \frac{2z+3}{z-4}$, z is a complex number.

(e) If W is a bilinear transformation and $W(z_i) = w_i$, $i = 1, 2, 3, 4$, then find $W\left(\frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)}\right)$, where z_i, for $i = 1, 2, 3, 4$ are complex numbers.

(f) Let C_1 and C_2 be two simple closed curves, then show that $\oint_{C_1} z\,dz = \oint_{C_2} z\,dz$.

(g) "If $f(z) = u(x, y) + iv(x, y)$ is analytic and u, v are real valued functions, then $u(x, y)$, $v(x, y)$ are harmonic." State whether the statement is true or not. Justify your answer.

2. Answer the following questions : $2 \times 4 = 8$

(a) Show that $z = \log\{(x-a)^2 + (y-b)^2\}$, satisfies $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$, except at (a, b).

(b) Prove that the function $f(x, y) = x^2 - 2xy + y^2 + x^4 + y^4$ has a minima at the origin.

(c) If P^* be a refinement of a partition P, then for a bounded function f, prove that $L(P^*, f) \geq L(P, f)$, symbols have their usual meanings.

(d) Evaluate $\left(\lim_{z \to 0} \frac{\bar{z}}{z}\right)$.

3. Answer any *three* parts : 5×3=15

(a) If V is a function of two variables x, y and $x = u\cos\alpha - v\sin\alpha$, $y = v\cos\alpha + u\sin\alpha$, where α is a constant, then show that

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 = \left(\frac{\partial V}{\partial u}\right)^2 + \left(\frac{\partial V}{\partial v}\right)^2$$

(b) Show that the improper integral $\int_a^b \frac{dx}{(x-a)^n}$ converges if and only if $n < 1$.

(c) Show that every absolutely convergent integral is convergent.

(d) Prove that if $f(z) = u(x, y) + iv(x, y)$, u and v are real valued functions, is analytic, then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

(e) Evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where C is the circle $|z| = 3$.

4. Answer *either* (a) *or* (b) :

(a) (i) Show that the limit exists at the origin but the repeated limits do not for the function $f(x, y)$, where

$$f(x, y) = \begin{cases} x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right), & xy \neq 0 \\ 0 & , \quad xy = 0 \end{cases}$$ 5

(ii) If $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, then find the maximum value of xyz. 5

(b) (i) Test the convergence of the integral

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{x^{p-1}} dx$$ 5

(ii) Prove that the improper integral $\int_a^b f\, dx$ converges if and only if to every $\varepsilon > 0$ there corresponds $\partial > 0$ such that

$$\left| \int_{a+\lambda_1}^{a+\lambda_2} f\, dx \right| < \varepsilon, \; 0 < \lambda_1, \, \lambda_2 < \delta$$ 5

5. Answer *either* (a) *or* (b) :

(a) (i) Prove that if a function f is continuous on $[a, b]$, then the function F, defined as $F(x) = \int_a^x f(t)\, dt$, $a \leq x \leq b$, is continuous and derivable on (a, b). 5

(ii) If a function f is bounded and integrable on $[a, b]$ and there exists a function F such that $F' = f$ on $[a, b]$, then prove that $\int_a^b f\,dx = F(b) - F(a)$. 5

(b) (i) Prove that if f is a non-negative continuous function on $[a, b]$ and $\int_a^b f\,dx = 0$, then $f(x) = 0$ for all $x \in [a, b]$. 5

(ii) Show that $f(x) = [x]$ is Riemann integrable on $[0, 2]$, where $[x]$ denotes the greatest integer not greater than x. 5

6. Answer *either (a) or (b)* :

(a) (i) Prove that $f(z) = \frac{1}{z^2}$ is not uniformly continuous in the region $|z| \le 1$ but uniformly continuous in the region $\frac{1}{2} \le z \le 1$. 5

(ii) Find the orthogonal trajectories of the family of curves in the xy-plane defined by $e^{-x}(x \sin y - y \cos y) = \alpha$, α is a real constant. 5

(b) (i) If $f(z)$ is analytic inside and on a circle C of radius r and center at $z = a$, then prove that $|f^{(n)}(a)| \leq M\frac{n!}{r^n}$, for $n = 0, 1, 2, 3, \ldots$ and M is a constant such that $|f(z)| < M$. 5

(ii) Find a bilinear transformation which maps z_1, z_2, z_3 of the z-plane into the points w_1, w_2, w_3 of w-plane respectively. 5

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