# 2017

## MATHEMATICS

## ( Major )

Paper : 5.5

## ( Probability )

*Full Marks* : 60

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

1. Answer the following questions as directed : $1 \times 8 = 8$

   (a) Two mutually exclusive events with positive probabilities are independent.

   (State whether the above statement is true or false)

   (b) Mention two properties which must be satisfied by the distribution function $F(x)$ for random variable $X$.

(c) If $X$ is a random variable with its mean $\overline{X}$, then the expression $E(X-\overline{X})^2$ represents

(i) the variance of $X$

(ii) second central moment

(iii) Both (i) and (ii)

(iv) None of (i) and (ii)

(Choose the correct option)

(d) The mean, mode and median of a continuous distribution are coincide. Name the distribution.

(e) Let $A$, $B$ and $C$ are three mutually exclusive and exhaustive events associated with a random experiment. Find—

$$P(A) \text{ if } P(B) = \frac{3}{2}P(A) \text{ and } P(C) = \frac{1}{2}P(B)$$

(f) If the probability of a defective bolt is 0·1, find the standard deviation for the number of defective bolts in a total of 400 bolts.

(g) Let $X$ be a random variable. Then for

$$f(x) = ke^{-2x}, \quad x \geq 0$$
$$= 0, \quad \text{otherwise}$$

to be density function, $k$ must be equal to

(i) 2

(ii) $\frac{1}{2}$

(iii) 0

(iv) 1 (Choose the correct option)

(h) State the relationship between the moment generating function of the sum of a number of independent random variables and the moment generating function of these individual random variables.

**2.** Answer the following questions : 3×4=12

(a) If $B \subset A$, then prove that—

(i) $P(A \cap \overline{B}) = P(A) - P(B)$;

(ii) $P(B) \leq P(A)$;

where $\overline{B}$ is complement of $B$. 2+1=3

(b) The distribution function of a random variable $X$ is

$$F(x) = 1 - e^{-2x}, \quad x \geq 0$$
$$= 0, \quad x < 0$$

Find—

(i) the density function;

(ii) $P(-3 < X \leq 4)$. 1+2=3

(c) If $X$ is a random variable, then show that the quantity $E[(X-a)^2]$ is a minimum when $a = \mu = E(X)$.

(d) Find the moment generating function of a random variable $X$ that is binomially distributed.

**3.** Answer any *two* parts from the following questions : 5×2=10

(a) A card is drawn at random from an ordinary deck of 52 cards. Let $A$ be the event (king is drawn) or simply (king) and $B$ the event (club is drawn) or simply (club). Describe the following events :

(i) $A \cup B$

(ii) $A \cap B$

(iii) $A' \cup B'$

(iv) $A - B$

(v) $A \cup B'$

(b) State and prove Bayes' theorem.

(c) Three balls are drawn successively from a box containing 6 red balls, 4 white balls and 5 blue balls. Find the probability that they are drawn in the order red, white and blue if each ball is *(i)* replaced and *(ii)* not replaced.

**4.** Answer any *two* parts from the following questions : 5×2=10

(a) Let $X$ and $Y$ be jointly distributed with probability density function

$$f_{XY}(x, y) = \frac{1}{4}(1+xy), \quad |x|<1, \ |y|<1$$
$$= 0 \quad , \quad \text{otherwise}$$

Show that $X$ and $Y$ are not independent but $X^2$ and $Y^2$ are independent.

(b) A random variable $X$ has the density function

$$f(x) = \frac{c}{x^2+1}, \quad -\infty < x < \infty$$

Find—

*(i)* the value of $c$;

*(ii)* the probability that $X^2$ lies between $\frac{1}{3}$ and 1. 2+3=5

*(c)* The joint probability of two discrete random variables $X$ and $Y$ is given by

$$f(x, y) = \frac{1}{42}(2x+y), \quad 0 \le x \le 2, \; 0 \le y \le 3$$

($x$ and $y$ can assume all integers)

$$= 0, \quad \text{otherwise}$$

Find—

*(i)* $f(y|2)$;

*(ii)* $P(Y=1 \mid X=2)$.

**5.** Answer any *two* parts from the following questions : 5×2=10

*(a)* Define covariance of two random variables.

If $X$ and $Y$ are two random variables, prove that

$$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2\,\text{cov}(X, Y)$$

1+4=5

*(b)* Let $X_1, X_2, \ldots, X_n$ be mutually independent random variables (discrete or continuous), each having finite mean $\mu$ and variance $\sigma^2$. Then if

$$S_n = X_1 + X_2 + \cdots + X_n \quad (n = 1, 2, \ldots)$$

Prove that $\lim_{n \to \infty} P\left(\left|\frac{S_n}{n} - \mu\right| \ge \varepsilon\right) = 0$

*(c)* A random variable $X$ has density function given by

$$f(x) = 2e^{-2x}, \quad x \geq 0$$
$$= 0 \quad , \quad x < 0$$

Find—

*(i)* the moment generating function;

*(ii)* the first four moments about the origin. 2+3=5

**6.** Answer any *two* parts from the following questions : 5×2=10

*(a)* If $X$ and $Y$ are independent Poisson variates such that $P(X=1) = P(X=2)$ and $P(Y=2) = P(Y=3)$, find the variance of $X-2Y$.

*(b)* Write the probability density function of a random variable $X$ which follows normal distribution with mean $\mu$ and variance $\sigma^2$. What is a standard normal variate? Find its mean and variance.

*(c)* Prove that the mean and variance of a binomially distributed random variable are respectively

$$\mu = np \quad \text{and} \quad \sigma^2 = npq$$

(where the symbols have their usual meanings).

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